Session 2 of Turbulence & Wind Energy

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Session - Man	Date	Content
1 - EP	07/12	RANS approach and models 1: generalities, $k-\epsilon$ model, begin. of pb. 1.1
→ 2 - EP	14/12	RANS approach and models 2: middle of pb 1.1
3 - EP	04/01	RANS approach and models 3: end of pb 1.1, pb 1.2, $k-\omega$ model
EP	11/01	**Exam 1:** RANS models
4 - MH	06/02	Wind resources - Conversion principles - Aero
5 - MH	07/02	Aero - Stochastic (Langevin) power curve
6 - MH	08/02	Wind field and Turbulence
EP & MH	08/02	**Exam 2:** Turbulence & Wind Energy
MH	08/02	**General conf.:** Turbulence & Wind Energy Research

- RANS = Reynolds Averaged Navier-Stokes
- During sessions 2 - 3 and **Exam 1**, with EP, you use **Matlab** on your laptop.
- During sessions 4 - 6 and **Exam 2**, with MH, you use **R** on your laptop.

http://emmanuelplaut.perso.univ-lorraine.fr/twe

RANS approach and eddy viscosity assumption

$$\underbrace{v_i}_{\text{velocity component}} = \underbrace{\overline{v_i}}_{\text{average } U_i} + \underbrace{u_i}_{\text{fluctuations}} \qquad \underbrace{p}_{\text{modified pressure}} = \underbrace{\overline{p}}_{\text{average } P} + \underbrace{p'}_{\text{fluctuations}}$$

into the **Navier-Stokes equation** for a newtonian incompressible fluid

$$\rho\left[\partial_t v_i + \partial_{x_j}(v_i v_j)\right] = \partial_{x_j}(\sigma_{ij}) = -\partial_{x_i} p + \partial_{x_j}(\tau_{ij})$$

with the Cauchy stress tensor $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$

where the viscous-stress tensor $\tau_{ij} = 2\eta S_{ij}(\mathbf{v})$,

η = dynamic viscosity, $S_{ij}(\mathbf{v}) = \frac{1}{2}\left[\partial_{x_j}(v_i) + \partial_{x_i}(v_j)\right]$ = strain-rate tensor

$\rightarrow$ **Reynolds equation**

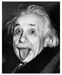
$$\rho\left[\partial_t U_i + \partial_{x_j}(U_i U_j)\right] = -\partial_{x_i} P + \partial_{x_j}(2\eta S_{ij}(\mathbf{V})) + \partial_{x_j}(\tau_{ij})$$

with the **Reynolds-stress tensor** $\tau_{ij} = -\rho\,\overline{u_i u_j} = -\rho$ covariance$(v_i v_j)$

Boussinesq EVA: $\exists$ **eddy viscosity** η_t s.t.

$$\tau_{ij} = -\frac{2}{3}\rho k \delta_{ij} + 2\eta_t S_{ij}(\mathbf{V})$$

with the **turbulent kinetic energy** (per unit mass) $k = \frac{1}{2}\overline{u_i u_i}$



RANS eddy viscosity models

→ **Reynolds-Boussinesq equation** or **RANS momentum equation**

$$\rho\left[\partial_t U_i + \partial_{x_j}(U_i U_j)\right] = -\partial_{x_i}\Big(P + \underbrace{\tfrac{2}{3}\rho k}_{\substack{\text{turbulent}\\ \text{Bernoulli}\\ \text{effect}}} \Big) + 2\partial_{x_j}\Big[\Big(\eta + \underbrace{\eta_t}_{\substack{\text{turbulent}\\ \text{mixing}\\ \text{or diffusion}}} \Big) S_{ij}(\mathbf{V})\Big]$$

with the **turbulent kinetic energy** $\boxed{k = \dfrac{1}{2}\overline{u_i u_i}}$

→ introduce the pressure with turbulence corrections $\Pi = P + \frac{2}{3}\rho k$.

RANS approach: mixing length model

→ **Reynolds-Boussinesq equation**

$$\partial_t U_i \ + \ \partial_{x_j}(U_i U_j) \ = \ -\frac{1}{\rho}\partial_{x_i}\Pi \ + \ 2\partial_{x_j}[(\nu+\nu_t)S_{ij}(\mathbf{V})]$$

with modified pressure Π and the **kinematic eddy viscosity**

$$\nu_t \ = \ \frac{\eta_t}{\rho} \ = \ ?$$

Prandtl's **mixing length model** considers **shear flows** where the mean strain-rate tensor

$$S_{ij}(\mathbf{V}) = \frac{1}{2}\left[\partial_{x_j}(U_i) + \partial_{x_i}(U_j)\right] \ \neq \ 0$$

as measured by

$$S \ = \ \sqrt{2S_{ij}S_{ij}} \ > \ 0 \ .$$

Idea: this mean strain-rate controls the turbulent mixing, therefore

$$\nu_t \ \propto \ S \ , \quad \nu_t \ \equiv \ v\,\ell \ \equiv \ \ell^2\,\frac{v}{\ell} \ = \ \ell_m^2\,S$$

with ℓ_m the **mixing length**... to be estimated !

RANS Launder & Spalding $k-\epsilon$ model

→ **Reynolds-Boussinesq equation**

$$\partial_t U_i \; + \; \partial_{x_j}(U_i U_j) \; = \; -\frac{1}{\rho}\partial_{x_i}\Pi \; + \; 2\partial_{x_j}[(\nu+\nu_t)S_{ij}(\mathbf{V})]$$

with modified pressure Π that implies the **turbulent kinetic energy**

$$\boxed{k \; = \; \frac{1}{2}\overline{u_i u_i}} \; .$$

It fulfills

$$\partial_t k \; + \; U_j \partial_{x_j} k \; = \; \partial_{x_j}(\nu \partial_{x_j} k) \; - \; \partial_{x_j}\left(\frac{1}{\rho}\overline{p' u_j} \; + \; \frac{1}{2}\overline{u_i u_i u_j}\right) + (\partial_{x_j} U_i)\frac{\tau_{ij}}{\rho} - \epsilon$$

with the **turbulent dissipation**

$$\boxed{\epsilon \; = \; \nu\overline{(\partial_{x_j} u_i)(\partial_{x_j} u_i)}} \; .$$

Idea: the **eddy viscosity** is controlled by **these two quantities**,
by dimensional analysis

$$\nu_t \; = \; \eta_t/\rho \; = \; C_\nu \; k^2/\epsilon$$

since $\nu_t \; \equiv \; v \; \ell \; , \quad k \; \equiv \; v^2 \; , \quad \epsilon \; \equiv \; v^3/\ell$.

Launder & Spalding $k-\epsilon$ model

- Equation for **V** : **Reynolds-Boussinesq equation**:

$$\partial_t U_i \;+\; \partial_{x_j}(U_i U_j) \;=\; -\frac{1}{\rho}\partial_{x_i}\Pi \;+\; 2\partial_{x_j}[(\nu+\nu_t)S_{ij}(\mathbf{V})]\;.$$

- Exact equation for k :

$$\partial_t k \;+\; U_j\partial_{x_j} k \;=\; \partial_{x_j}(\nu\partial_{x_j} k) \;-\; \underbrace{\partial_{x_j}\left(\frac{1}{\rho}\overline{p' u_j} \;+\; \frac{1}{2}\overline{u_i u_i u_j}\right)}_{\text{unknown } \Rightarrow \text{ modeled with } \nu_t} \;+\; P_k \;-\; \epsilon$$

with the **production term** deduced from the EVA

$$P_k \;=\; (\partial_{x_j} U_i)\frac{\tau_{ij}}{\rho} \;=\; 2\nu_t S_{ij}(\mathbf{V})S_{ij}(\mathbf{V}) \;>\; 0$$

Model equation:

$$\partial_t k \;+\; U_j\partial_{x_j} k \;=\; \partial_{x_j}[(\nu+\nu_t)\partial_{x_j} k] \;+\; P_k \;-\; \epsilon\;.$$

Launder & Spalding $k-\epsilon$ model

- Equation for $\mathbf{V}$: **Reynolds-Boussinesq equation**:

$$\partial_t U_i \ + \ \partial_{x_j}(U_i U_j) \ = \ -\frac{1}{\rho}\partial_{x_i}\Pi \ + \ 2\partial_{x_j}[(\nu+\nu_t)S_{ij}(\mathbf{V})] \ .$$

- Equation for k :

$$\partial_t k \ + \ U_j \partial_{x_j} k \ = \ \partial_{x_j}[(\nu+\nu_t)\partial_{x_j} k] \ + \ P_k \ - \ \epsilon$$

with the **production term** $P_k \ = \ 2\nu_t S_{ij}(\mathbf{V})S_{ij}(\mathbf{V})$.

- Equation for ϵ obtained by analogy with the k equation + dimensional analysis:

$$\partial_t \epsilon \ + \ U_j \partial_{x_j}\epsilon \ = \ \partial_{x_j}[(\nu+\sigma_\epsilon^{-1}\nu_t)\partial_{x_j}\epsilon] \ + \ \frac{\epsilon}{k}(C_1 P_k \ - \ C_2\epsilon) \ .$$

Launder & Spalding $k-\epsilon$ model

$$\frac{DU_i}{Dt} = \partial_t U_i + U_j \partial_{x_j} U_i = -\frac{1}{\rho}\partial_{x_i}\Pi + 2\partial_{x_j}[(\nu+\nu_t)S_{ij}(\mathbf{V})]$$

$$\frac{Dk}{Dt} = \partial_t k + U_j \partial_{x_j} k = \partial_{x_j}[(\nu+\nu_t)\partial_{x_j} k] + P_k - \epsilon$$

$$\frac{D\epsilon}{Dt} = \partial_t \epsilon + U_j \partial_{x_j} \epsilon = \partial_{x_j}[(\nu+\sigma_\epsilon^{-1}\nu_t)\partial_{x_j}\epsilon] + \frac{\epsilon}{k}(C_1 P_k - C_2 \epsilon)$$

$$P_k = 2\nu_t S_{ij}(\mathbf{V})S_{ij}(\mathbf{V}) , \quad \nu_t = C_\nu \, k^2/\epsilon$$

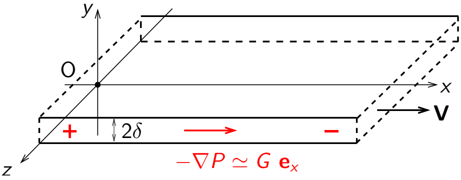
Coefficient	σ_ϵ	C_1	C_2	C_ν	Von Karman constant κ
'Universal' (?) standard value	1.3	1.44	1.92	0.09	0.43

- In a **plane jet** or **mixing layer**, the rate of growth of the flow $\rightarrow$ C_1
- In **turbulence behind a grid**, the rate of decay of the flow $\rightarrow$ C_2
- In the **overlap layer**, **wall laws** with a κ value $\rightarrow$ C_ν and σ_ϵ

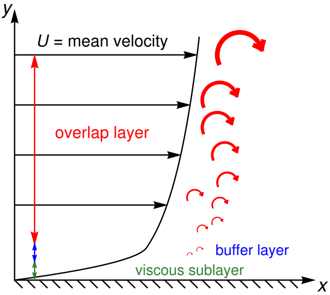
[Launder & Spalding 1974 *Comp. Meth. Appl. Mech. Eng.*]

Pb 1.1: learn about turbulent flows, the mixing-length and $k - \varepsilon$ RANS models, by focussing on channel flows for which there are recent DNS

General setup:



Focus near the lower wall:



Mean velocity $\mathbf{V} = U(y)\ \mathbf{e}_x$, mean (modified) pressure $P = -Gx + p_0(y)$,

all other mean fields depend only on y.

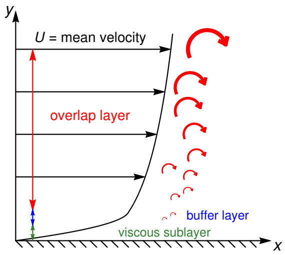
Pb 1.1: part 1: generalities and mixing-length model: q. 1.1 to 1.6

- Integrated RANS momentum eq. in the x-direction :

$$\eta \frac{dU}{dy} + \tau_{xy} = \tau_w - Gy$$

with the **mean wall shear stress**

$$\tau_w = \eta \left.\frac{dU}{dy}\right|_{y=0} = \eta\, U_0' = G\,\delta\ .$$



- **Viscous sublayer**: y small, viscous stress dominates

$$\eta \frac{dU}{dy} \simeq \tau_w \quad \Longrightarrow \quad U \simeq U_0' y$$

$$\Longleftrightarrow \quad U^+ = \frac{U}{u_\tau} = y^+ = \frac{y u_\tau}{\nu}$$

with u_τ the **friction velocity** s.t. $\tau_w = \rho u_\tau^2$.

Pb 1.1: part 1: generalities and mixing-length model: q. 1.1 to 1.9

- Integrated RANS momentum eq. in the x-direction :

$$\eta \frac{dU}{dy} + \tau_{xy} = \tau_w - Gy$$

with the **mean wall shear stress**

$$\tau_w = \eta \left. \frac{dU}{dy} \right|_{y=0} = \eta U_0' = G \delta .$$

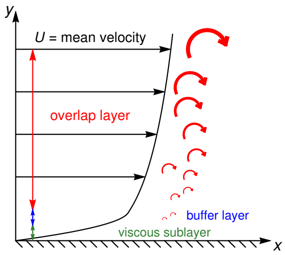
- **Log layer**: y larger, Reynolds stress dominates, estimated with a **mixing-length model** using $\ell_m = \kappa y$

$$\Longrightarrow \tau_{xy} \simeq \eta_t \frac{dU}{dy} \simeq \tau_w , \quad \eta_t \simeq \rho \frac{dU}{dy} (\kappa y)^2$$

$$\Longrightarrow U^+ = \frac{U}{u_\tau} = \frac{1}{\kappa} \ln y^+ + C$$

with u_τ the **friction velocity** s.t. $\tau_w = \rho u_\tau^2$.

NB: $\eta_t = \rho u_\tau \kappa y \iff \nu_t = u_\tau \kappa y \iff \nu^+ := \frac{\eta_t}{\eta} = \kappa y^+$.



Part 2: comparisons with the UT DNS regarding the viscous sublayer

2 With the DNS files of the UT database

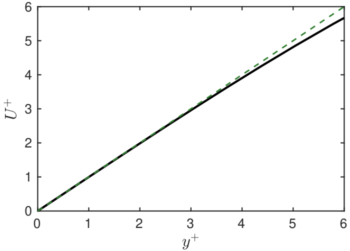
https://turbulence.oden.utexas.edu

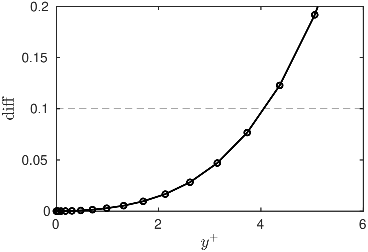
using the **Matlab** script in the lecture notes,
evidence and characterize the **viscous sublayer** for $Re_\tau = 180$ and 5200...

Part 2: comparisons with the UT DNS regarding the viscous sublayer

2 DNS data at $Re_\tau = 180$ display a **viscous sublayer** for $y^+ \lesssim y_0^+ = 4$ with a criterion

$$\text{diff} = |U^+ - y^+| < 10\%$$

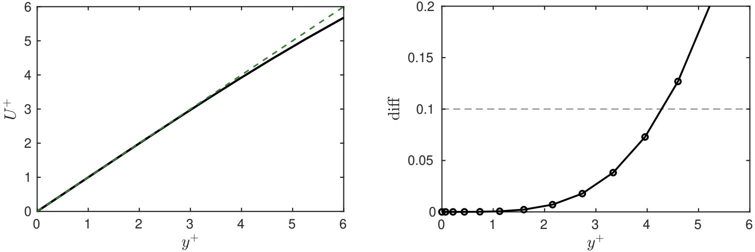




Part 2: comparisons with the UT DNS regarding the viscous sublayer

2 DNS data at $Re_\tau = 5200$ display a **viscous sublayer** for $y^+ \lesssim y_0^+ = 4$ with a criterion

$$\text{diff} = |U^+ - y^+| < 10\%$$



NB: there are less grid points at $Re_\tau = 5200$ - why ?