Exact eddy-viscosity equation for turbulent wall flows

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Table of contents

What's new in this Version 0.015 of April 29, 2020

- I have inserted the answers of Stefan (mail of April 28, 2020) to Q1 and Q2 in the sections 3.1 and 3.2.

Abstract

A recent theory has been developed (Heinz 2018, 2019) for three canonical turbulent wall flows: channel flow, pipe flow and zero-pressure gradient boundary layer, that offers exact analytical formulas for the RANS eddy-viscosity. By calculating the eddy-viscosity turbulent diffusion term for these flows where the turbulence is stationary, one identifies a RANS eddy-viscosity equation with one production and two dissipation terms. One dissipation term is universal and peaks in the near-wall region. The second one is flow-dependent and peaks in the wake region. The production term is flow-dependent and peaks in between. The universal dissipation term implies a damping function and a length scale analogous to the von Karman length scale used in the Scale-Adaptative Simulation models.

...To be expanded ?..

1 Introduction

To be written !..

2 State of the art

2.1 Models with an eddy-viscosity equation

To be written, by citing at least Nee & Kovasznay (1969); Spalart & Allmaras (1994); Baldwin & Barth (1990); Menter (1997); Yoshizawa *et al.* (2012) *!..*
Maybe, already at the level where we speak of Nee & Kovasznay (1969)*, introduce notations for* 2D xy flows and boundary layer coordinates, *e.g. the definitions of* ν, u_τ , y^+, U^+, the dimensionless mean strain rate

$$S^+ \;=\; dU^+/dy^+ \;, \tag{1}$$

and the eddy viscosity

$$\nu_t = -\langle u_x u_y\rangle / (dU/dy) \;?.. \tag{2}$$

...These definitions may be placed somewhere else ?..

2.2 The Scale - Adaptive Simulation models

To be written, by citing at least Menter & Egorov (2006, 2010); Egorov *et al.* (2010); Abdol-Hamid (2015) *!..*
Introduce in particular the von Karman length-scale

$$L_{vK} \;=\; \kappa \left| \frac{dU/dy}{d^2U/dy^2} \right| \tag{3}$$

and κ the von Karman constant *!..*

2.3 Analytic eddy viscosity model of turbulent wall flows

Heinz (2018, 2019) proposed analytic models for the mean flow, main Reynolds stress and eddy viscosity of three canonical wall-bounded turbulent flows: channel flow, pipe flow and the zero-pressure gradient turbulent boundary layer. These flows present a certain richness, and are denoted here 'turbulent wall flows'. In the equation (11) of Heinz (2019), an analytic expression is proposed for the reduced eddy viscosity $\nu^+ = \nu_t/\nu$, which is valid at high Reynolds number, $Re_\tau \gtrsim 500$,

$$\nu^+ \;=\; (1/S_{12}^+ - 1)\; N \;. \tag{4}$$

There $S_{12}^+ = S_1^+ + S_2^+$ is a very good approximation of the dimensionless mean strain rate S^+ (1) in the inner region of the flows, i.e., disregarding wake effects, see the equation (7) of Heinz (2018) and the corresponding discussion. Precisely

$$S_{12}^+ \;=\; S_{12}^+(y^+) \;=\; 1 \;-\; \left[\frac{(y^+/a)^{b/c}}{1+(y^+/a)^{b/c}}\right]^c \;+\; \frac{1}{\kappa y^+}\,\frac{1+h_2/(1+y^+/h_1)}{1+y_k/(y^+H)} \;, \tag{5}$$

with

$$a=9,\;\; b=3.04,\;\; c=1.4,\;\; H \;=\; H(y^+) \;=\; (1+h_1/y^+)^{-h_2} \;,\;\; h_1=12.36,\; h_2=6.47,\; y_k=75.8 \;, \tag{6}$$

and the von Karman constant

$$\kappa \;=\; 0.40 \;. \tag{7}$$

The universal function S_{12}^+ , plotted on the figure 1a, approaches naturally 1 as $y^+ \to 0$ in the viscous sublayer. On the contrary, as $y^+ \to \infty$, $S_{12}^+ \sim 1/(\kappa y^+)$, in agreement with the log law. Therefore the function $1/S_{12}^+ - 1$, plotted on the figure 1b, which appears in the eddy viscosity (4), vanishes in the limit $y^+ \to 0$, and then increases smoothly to approach the function κy^+ as $y^+ \to \infty$.
The second ingredient of the theory is the function N, which is flow-dependent and written in outer scaling, because it describes wake effects. Indeed, with the notations of Heinz (2018, 2019), $N = 1/G_{CP}$ for channel and pipe flows, M_{BL}/G_{BL} for boundary layers, where G_{CP} and G_{BL} characterize the wake contribution S_3^+ to the dimensionless mean strain rate S^+ (see the equations 7 and A.22 of Heinz 2018), M_{BL} characterizes the total stress in boundary layers (see the equation 4 of Heinz 2019). Thus, for channel and pipe flows

$$N \;=\; N_X(y/\delta) \quad\text{with}\quad N_X(y) \;=\; \frac{K_X y + (1-y)^2(0.6y^2+1.1y+1)}{1+y+y^2(1.6+1.8y)} \;, \tag{8}$$

$X=C,\; K_C = 0.933$ for channel, $X=P,\; K_P=0.687$ for pipe; for boundary layers

$$N \;=\; N_{BL}(y/\delta) \quad\text{with}\quad N_{BL}(y) \;=\; \frac{1+0.285\; y\; e^{y(0.9+y+1.09y^2)}}{1+(0.9+2y+3.27y^2)y} \;\; e^{-y^6-1.57y^2} \;. \tag{9}$$

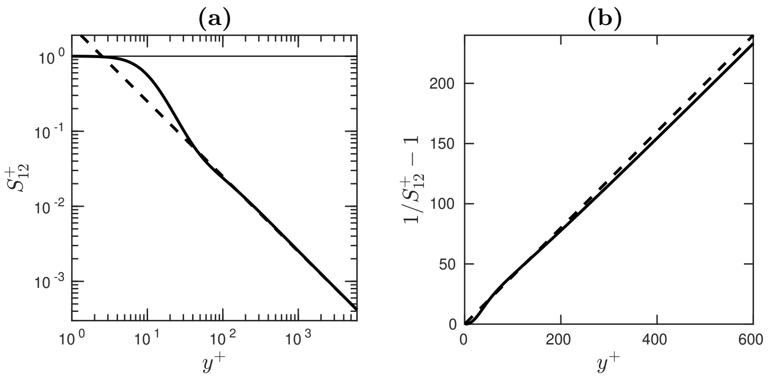

Fig. 1 : **(a)** Continuous line: S_{12}^+ , dashed line: $1/(\kappa y^+)$. **(b)** Continuous line: $1/S_{12}^+ - 1$, dashed line: κy^+.

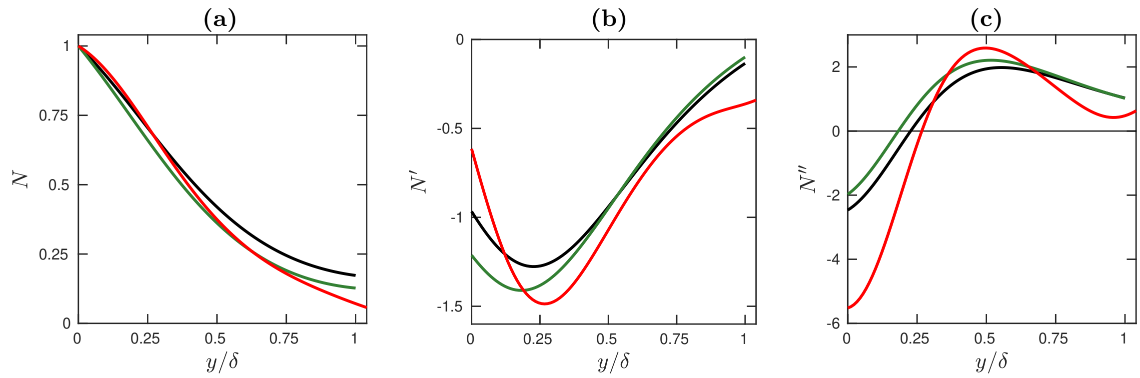

Fig. 2 : **(a)** N **(b)** N' **(c)** N'' for channel (black), pipe (green), boundary layer (red).

The length scale δ is the half-channel height, pipe radius, or 99% boundary layer thickness with respect to channel flow, pipe flow, and boundary layer, respectively. The function N is plotted for these three flows on the figure 2a. In the near-wall region, when $y/\delta \to 0$, $N \to 1$, hence the eddy viscosity (4), $\nu^+ = (1/S_{12}^+(y^+) - 1)\ N(y^+/\delta^+) \sim (1/S_{12}^+(y^+) - 1)$ where $\delta^+ = Re_\tau$. Hence the log-law eddy viscosity κy^+ is approximately recovered if $1 \ll y^+ \ll \delta^+$; for a more precise study, see the section 4.1 of Heinz (2019). When y becomes of the order of δ, wake effects come in, that saturate the growth of the eddy viscosity (4), since N decreases. Whereas the maximum value of y is δ in channel and pipe flows (in channel flow if $y \in [\delta, 2\delta]$ the mean fields can be obtained by suitable symmetries from the mean fields for $y \in [0, \delta]$), it may be much larger in boundary layers. Naturally, $N_{BL} \to 0$ as $y \to \infty$; precisely $N_{BL} < 10^{-3}$ as soon as $y > 1.36\delta$.

The theory of Heinz (2018, 2019) has been validated by a thorough study of DNS data, including those of Lee & Moser (2015), and experimental results. For instance, the figures S.6abc of the supplementary material to Heinz (2019) show the eddy-viscosity of various DNS, one for each canonical flow, compared with two variants of the eddy-viscosity model (4). In particular, the magenta curves show $\kappa y^+ N$ with our notations, i.e. $(1/S_{12}^+ - 1)$ in (4) has been replaced by κy^+. The agreement with the DNS is good, except in the outer region, where in (2) both the numerator $\langle u_x u_y \rangle$ and the denominator dU/dy tend to zero, hence the DNS noise is amplified.

Since the derivatives N' and N'' will be needed hereafer, they are plotted on the figures 2bc. Whereas the functions N for the three flow cases are quite similar (figure 2a), their first and second derivatives show larger differences (figures 2bc).

3 Analysis: exact eddy-viscosity equation

3.1 Q1 regarding near-wall effects / the kinematic viscosity

- **Q1 :** We agreed that our model should work as soon as $y^+ \gtrsim 1$, which is still in the viscous sublayer.
  We know that y^+ has to reach values of the order of 33 to have $\nu^+ > 10$ i.e. '$\nu^+ \gg 1$'. Thus, in the range $y^+ \in [1, 33]$ where our model should operate, the kinematic fluid viscosity is ***not*** negligible in front of the eddy viscosity.
  Taking this effect into account means that the dimensionless ν_t turbulent diffusion term should not be

  $$D_\nu^+ = \frac{\partial}{\partial y^+}\left(\nu^+ \frac{\partial \nu^+}{\partial y^+}\right) \tag{10}$$

  but

  $$D_\nu^+ = \frac{\partial}{\partial y^+}\left((1+\nu^+) \frac{\partial \nu^+}{\partial y^+}\right) , \tag{11}$$

  and there may even be the question of adding a factor σ of order 1 in front of ν^+ in $(1 + \nu^+)$.

  I suggest to disregard this and not mention this, i.e. to work (though y^+ may be 'small') with the ***'high-Reynolds number'*** expression (10).

  Do you agree ?

- **R1 :** I agree.
  We should argue that we focus on $500 < Re_\tau$ up to infinity. No-one else was able to tackle the extreme Re regime before. Later, we may consider to look at this effect in an appendix, it should not be too bad.

3.2 Q2 regarding pipe flow

- **Q2 :** I am concerned with the form of the (high-Reynolds number !) ν_t turbulent diffusion term in pipe flow. From a theoretical point of view, we want a ν_t - equation that is intrinsic and independent of the system of coordinates chosen. Therefore, to me, the diffusion term should be, intrinsically,

  $$D_\nu = \mathrm{div}(\nu_t \, \mathbf{grad}(\nu_t)) . \tag{12}$$

  In cartesian coordinates, if ν_t depends only on y, this gives the classical

  $$D_\nu = \frac{\partial}{\partial y}\left(\nu_t \frac{\partial \nu_t}{\partial y}\right) . \tag{13}$$

  However, in cylindrical coordinates (r, θ, z), this gives, if ν_t depends only on r,

  $$D_\nu = \frac{1}{r}\frac{\partial}{\partial r}\left(r \nu_t \frac{\partial \nu_t}{\partial r}\right) . \tag{14}$$

  Since $y = \delta - r$ with δ the pipe radius,

  $$D_\nu = \frac{1}{\delta - y}\frac{\partial}{\partial y}\left((\delta - y) \nu_t \frac{\partial \nu_t}{\partial y}\right) \tag{15}$$

  which is the same as (13) only if $y \ll \delta$.

  I suggest to disregard this and not mention this, i.e. to work with the ***'cartesian'*** expressions (10) or (13).

  Do you agree ?

- **R2 :** I totally agree.

4 Discussion

To be written !..

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