Transition to turbulence in thermoconvection & aerodynamics

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Session	Date	Content
1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
2 -	06/10	**RB** Thermoconvection: linear stability analysis
3 -	13/10	**RB** Thermoconvection: (weakly) nonlinear phenomena
4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
→ 6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB = Rayleigh-Bénard **OSF** = Open Shear Flows

Today: session 6: transition in open shear flows:

- End of the linear analysis of TS waves in plane Poiseuille flow (PPF)
- Weakly nonlinear analysis of TS waves in PPF
- Openings: strongly nonlinear phenomena - transition in boundary layers

When and how 2D xz laminar open shear flows get unstable ?
General example: plane parallel flows

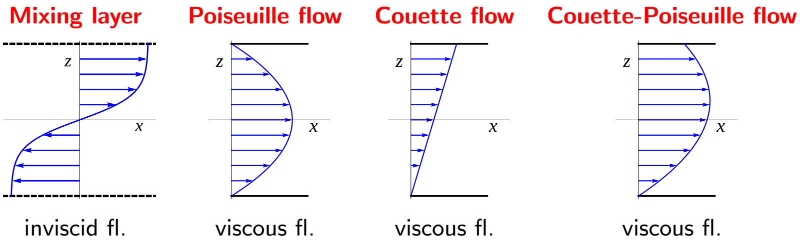
$$\mathbf{v} = \mathbf{v}_0 = U(z)\,\mathbf{e}_x\ ,\quad p = p_{\text{static}} + \rho g Z = 0 \text{ in an inviscid fluid,}$$
$$p = p_{\text{static}} + \rho g Z = -Gx \text{ in a viscous fluid,}$$

is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

$$\rho\big[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big] = -\boldsymbol{\nabla} p + \eta \Delta \mathbf{v}$$
$$\Longleftrightarrow \quad \mathbf{0} = G\mathbf{e}_x + \eta U''(z)\mathbf{e}_x$$

whatever $U(z)$ in an inviscid fluid,

provided $U(z) = \alpha + \beta z + \gamma z^2$ in a viscous fluid.



Stability analysis of plane parallel flows

Basic flow:

$$\mathbf{v}_0 \;=\; U(z)\;\mathbf{e}_x\;, \quad p_0 \;= -Gx \quad \text{with} \quad G = 0 \text{ in an inviscid fluid,}$$
$$G > 0 \text{ in a viscous fluid.}$$

Basic flow with **perturbations**:

$$\mathbf{v} \;=\; \mathbf{v}_0 \;+\; \mathbf{u}\;, \quad p \;=\; p_0 \;+\; \widetilde{p}$$

$$\partial_t \mathbf{v} \;+\; (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \;=\; -(1/\rho)\boldsymbol{\nabla} p \;+\; \nu\Delta\mathbf{v} \qquad \text{(NS)}$$

$$\partial_t \mathbf{u} \;+\; U' u_z \mathbf{e}_x \;+\; U\partial_x \mathbf{u} \;+\; (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -(1/\rho)\boldsymbol{\nabla}\widetilde{p} \;+\; \nu\Delta\mathbf{u} \qquad \text{(NS)}$$

$$\mathrm{div}\mathbf{v} \;=\; \mathrm{div}\mathbf{u} \;=\; 0 \qquad \text{(MC)}$$

▷ Unit of length $=\; h$ half-width of the channel, thickness of the mixing layer...
▷ Unit of velocity $=\; U_0 \;=\; \max_z U(z)$ scale of U
▷ Unit of time $=\; h/U_0$ advection time

$$\partial_t \mathbf{u} \;+\; U' u_z \mathbf{e}_x \;+\; U\partial_x \mathbf{u} \;+\; (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -\boldsymbol{\nabla}\widetilde{p} \;+\; R^{-1}\,\Delta\mathbf{u} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R \;=\; U_0 h/\nu}$, $R \;=\; \infty$ in an inviscid fluid.

2D xz stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U \partial_x \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \widetilde{p} + R^{-1} \Delta \mathbf{u} \, , \tag{NS}$$

$$\mathrm{div} \mathbf{u} = 0 \, . \tag{MC}$$

2D xz **perturbations** can be defined by their **streamfunction** $\psi(x,z)$:

$$\mathbf{u} = \mathbf{curl}(\psi \, \mathbf{e}_y) = (\nabla \psi) \times \mathbf{e}_y = -(\partial_z \psi) \, \mathbf{e}_x + (\partial_x \psi) \, \mathbf{e}_z \, .$$

We can eliminate $\widetilde{p}$ in (NS) by considering **curl**(NS) $\cdot \, \mathbf{e}_y$ i.e. the **vorticity equation**:

$$\partial_t(-\Delta\psi) + \left[\partial_z\left(\mathbf{u}\cdot\nabla u_x\right) - \partial_x\left(\mathbf{u}\cdot\nabla u_z\right)\right] = R^{-1}\Delta(-\Delta\psi) + U\partial_x(\Delta\psi) - U''(\partial_x\psi) \tag{Vort}$$

$$\iff \boxed{D \cdot \partial_t \psi = L_R \cdot \psi + N_2(\psi,\psi)} \, . \tag{Vort}$$

Boundary conditions:

viscous fluid : $\mathbf{u} = \mathbf{0} \iff \partial_x \psi = \partial_z \psi = 0$ if $z = z_\pm$,

inviscid fluid : $u_z = 0 \iff \partial_x \psi = 0$ if $z = z_\pm$.

2D xz linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \;=\; -\Delta \partial_t \psi \;, \quad L_R \cdot \psi \;=\; R^{-1} \Delta(-\Delta\psi) \;+\; U \partial_x(\Delta\psi) - U''(\partial_x \psi) \;,$$

$$\begin{array}{lcccl} \text{viscous fluid :} & \mathbf{u} = \mathbf{0} & \Longleftrightarrow & \partial_x \psi \;=\; \partial_z \psi \;=\; 0 & \text{if} \quad z = z_\pm \;, \\ \text{inviscid fluid :} & u_z = 0 & \Longleftrightarrow & \partial_x \psi \;=\; 0 & \text{if} \quad z = z_\pm \;. \end{array}$$

Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t) \;=\; \Psi_n(z)\; \exp[ik(x - c_r t)]\; \exp(k c_i t)$$

with $k \;=$ **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
$\sigma \;=$ **temporal eigenvalue**.

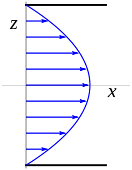
Most often the bulk velocity of the basic flow $\langle U \rangle_z > 0 \quad \Rightarrow \quad$ by advection

$\sigma \;=\; -i\omega \;=\; -ikc \quad$ with $\quad c$ the **complex phase velocity**,
$c_r > 0$ the **real phase velocity**,
$kc_i > 0$ (resp. < 0) the **growth rate** (resp. the opposite of the **damping rate**).

Stability of inviscid plane Poiseuille flow

According to the Rayleigh's criterion (ex 2.1),
plane Poiseuille flow of an **inviscid fluid** has no inflection point $\Rightarrow$ it is **stable**.

$$\mathbf{v}_0 = U_0(1-(z/h)^2)\ \mathbf{e}_x$$

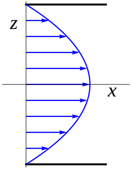


Stability of viscous plane Poiseuille flow

Pb 2.1:

plane Poiseuille flow of a **viscous fluid** may be **unstable** !

$$\mathbf{v}_0 = (1 - z^2)\ \mathbf{e}_x$$



Must calculate normal modes

$$\psi = \Psi(z)\ \exp(ikx + \sigma t) = \Psi(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

by solving the **vorticity equation**

$$\boxed{\sigma D\psi = -\sigma\Delta\psi = L_R\psi = -R^{-1}\Delta\Delta\psi + ik(U\Delta\psi - U''\psi)}$$

with the BC at $z = \pm 1$: $\Psi = \partial_z\Psi = 0.$

Eigenvalue $\sigma = -ikc$; $c_r = -\sigma_i/k$ phase velocity ; $\sigma_r > 0 \leftrightarrow$ **amplified mode**

$\sigma_r = 0 \leftrightarrow$ **neutral mode**

$\sigma_r < 0 \leftrightarrow$ **damped mode**

Stability of viscous plane Poiseuille flow: pb 2.1

$$\boxed{\sigma D\Psi \;=\; -\sigma\Delta\Psi \;=\; L_R\Psi \;=\; -R^{-1}\Delta\Delta\Psi \;+\; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(Vort)}$$

$$\text{with} \quad \Delta \;=\; -k^2 + \frac{d^2}{dz^2}$$

and the boundary conditions $\Psi \;=\; \Psi' \;=\; 0 \quad \text{if} \quad z = \pm 1$.

Spectral expansion taking into account the BC and even symmetry under $z \mapsto -z$:

$$\Psi(z) \;=\; \sum_{n=1}^{N} \Psi_n F_n(z)$$

$$\text{with} \quad F_n(z) \;=\; (z-1)^2\,(z+1)^2\;T_{2n-2}(z) \;=\; (z^2-1)^2\;T_{2n-2}(z)\;,$$

$$T_n(z) \;=\; n^{th} \text{ Chebyshev polynomial of the first kind.}$$

Evaluate (Vort) at the **Gauss-Lobatto collocation points**

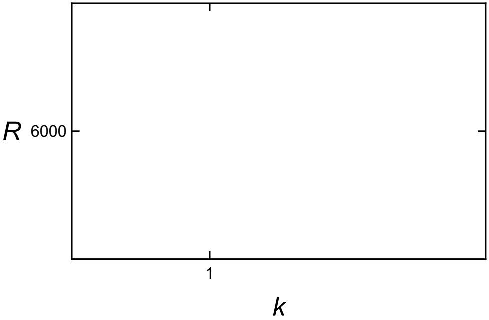
$$z_m \;=\; \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \{1,2,\cdots,N\}$$

$$\iff \quad \sigma\sum_n \Psi_n DF_n(z_m) \;=\; \sum_n \Psi_n LF_n(z_m) \quad \iff \quad \boxed{\sigma MD\cdot V \;=\; ML\cdot V}$$

$$\text{with} \quad V \;=\; (\Psi_1, \ldots, \Psi_N)^T \quad , \quad [MD]_{mn} \;=\; DF_n(z_m) \quad , \quad [ML]_{mn} \;=\; LF_n(z_m)\;.$$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:

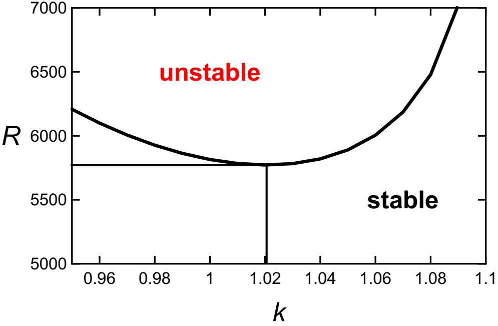


converged, near the critical k corresponding to the minimal R, within 0.1% provided that

$$Nz \geq 17?, 18?, 19?$$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:



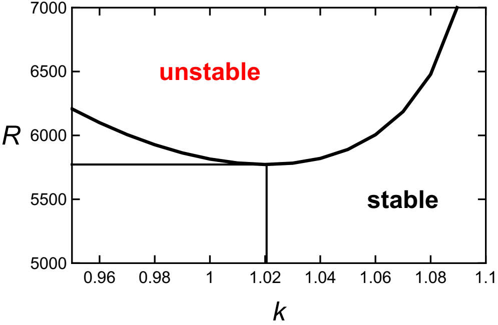
converged, near the critical k corresponding to the minimal R, within 0.1% provided that

$$Nz \geq 18$$

which is rather 'low': here the spectral method is quite efficient !

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:

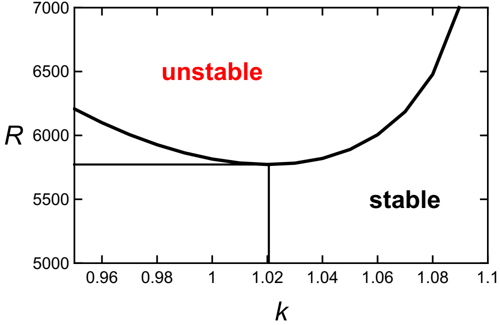


Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves

- **critical wavenumber** $k_c =$
- **critical Reynolds number** $R_c =$
- **critical angular frequency** $\omega_c =$ $\leftrightarrow$ **critical phase velocity** $c_c =$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:

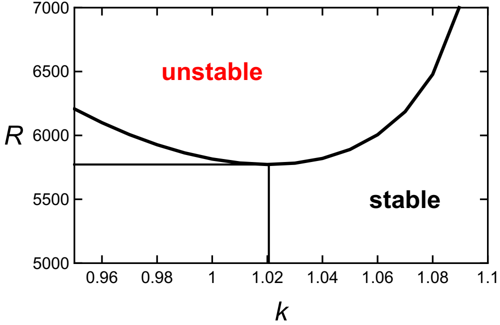


Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves

- **critical wavenumber** $k_c = 1.02$
- **critical Reynolds number** $R_c =$
- **critical angular frequency** $\omega_c =$ $\leftrightarrow$ **critical phase velocity** $c_c =$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:

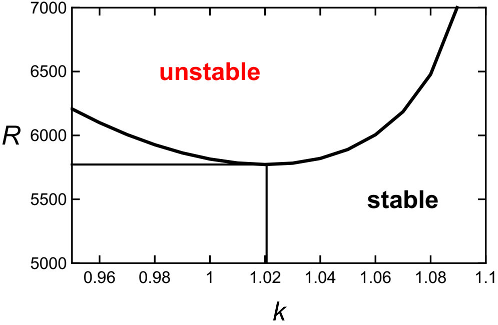


Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves

- **critical wavenumber** $k_c = 1.02$
- **critical Reynolds number** $R_c = 5772$
- **critical angular frequency** $\omega_c =$ $\leftrightarrow$ **critical phase velocity** $c_c =$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:

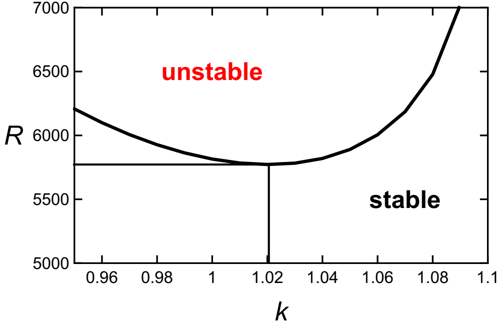


Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves

- **critical wavenumber** $k_c = 1.02$
- **critical Reynolds number** $R_c = 5772$
- **critical angular frequency** $\omega_c = 0.269 \leftrightarrow$ **critical phase velocity** $c_c =$

Stability of viscous plane Poiseuille flow: pb 2.1

Neutral curve:



Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves

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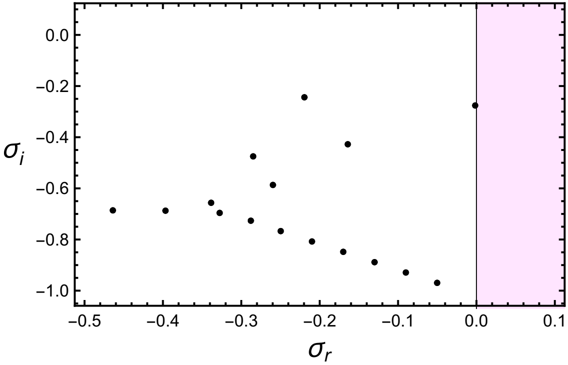
- **critical Reynolds number** $R_c = 5772$
- **critical angular frequency** $\omega_c = 0.269 \leftrightarrow$ **critical phase velocity** $c_c = 0.264$

Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue σ passes the real axis as R increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02 \; , \quad R = 5000$$

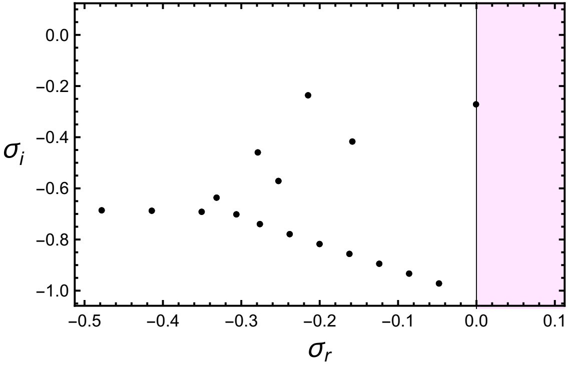


computed with $Nz = 40$ spectral modes and high precision numerics, see ex 2.2 (collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue σ passes the real axis as R increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02\ , \quad R = 5500$$



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Stability of viscous plane Poiseuille flow

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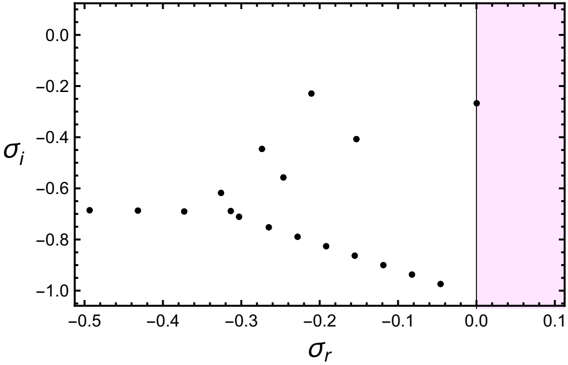
$$k = 1.02\ , \quad R = 6000$$



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Stability of viscous plane Poiseuille flow

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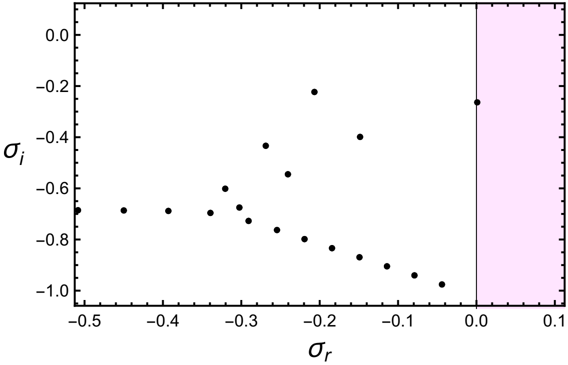

computed with $Nz = 40$ spectral modes and high precision numerics, see ex 2.2 (collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

Stability of viscous plane Poiseuille flow: pb 2.1

7.a Find the eigenvector of the spectral coefficients

$$V \ = \ (\Psi_1, \, ..., \Psi_N)^T$$

that represents the **critical mode** → calculate the **critical streamfunction**

$$\Psi(z) \ = \ \sum_{n=1}^{N} \Psi_n F_n(z)$$

→ normalize it s.t. $\Psi(z=0) \ = \ 1$ → plot its modulus vs z :

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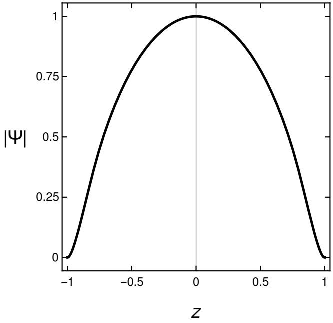

Stability of viscous plane Poiseuille flow: pb 2.1

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→ plot real & imaginary parts:

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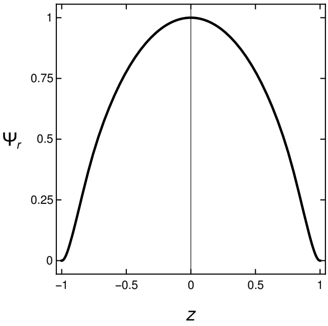
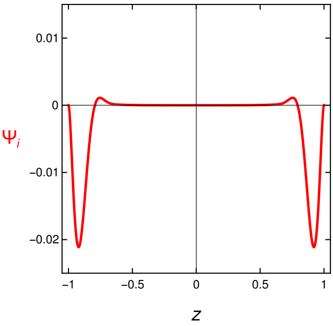
→ plot real & imaginary parts: $\Psi_i \neq 0$ creates interesting nonlinear effects, see ex 2.4,5 !





Stability of viscous plane Poiseuille flow: pb 2.1

7.b Save the normalized vector of the spectral coefficients $V = (\Psi_1, ..., \Psi_N)^T$ to a file `V1`.

8 In the *xz* plane, **streamlines** i.e. contour plots of the full streamfunction
$\Psi_0 + [A\ \Psi(z)\ \exp(ik_c x) + c.c.]$ with Ψ_0 the one of the basic flow,

for $A = 0$:

$A = 0.1$:

$A = 0.2$:

Stability of viscous plane Poiseuille flow: pb 2.1

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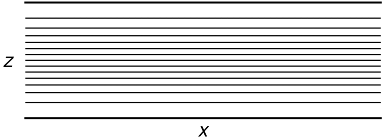

see Reynolds (1895) : motion is 'direct'

$A = 0.1$:

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Stability of viscous plane Poiseuille flow: pb 2.1

7.b Save the normalized vector of the spectral coefficients $V = (\Psi_1, ..., \Psi_N)^T$ to a file `V1`.

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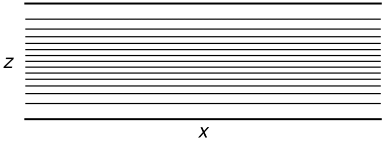

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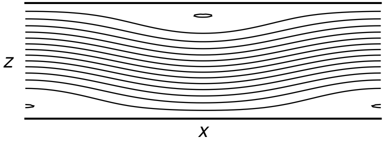
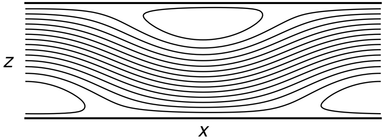


motion is 'sinuous' !

$A = 0.2$:



motion is 'sinuous' !

Towards the weakly nonlinear analysis...

Like in 2D RBT, we use the **basis of the linear modes**.

Work in a box with periodic BC under $x \mapsto x + \lambda_c \;\Rightarrow\;$ the wavenumber $k \in \mathbb{K} = k_c\mathbb{Z}$.

A general streamfunction

$$\psi \;=\; \sum_{k\in\mathbb{K}} \sum_{n\in\mathbb{N}^*} A(k,n)\;\psi_1(k,n) \;=\; \sum_{\mathbf{q}} A(\mathbf{q})\;\psi_1(\mathbf{q}) \quad \text{with} \quad \mathbf{q} \;=\; (k,n) \in \mathbb{K}\times\mathbb{N}^*\;.$$

To calculate the amplitudes $A(\mathbf{q})$...

$$\Longrightarrow \quad \text{the amplitude of the critical mode} \;\boxed{A(\mathbf{q}_c) \;=\; \langle D\cdot\psi,\, \phi_{1c}\rangle}\;.$$

Towards the weakly nonlinear analysis...

Like in 2D RBT, we use the **basis of the linear modes**.

Work in a box with periodic BC under $x \mapsto x + \lambda_c \;\Rightarrow\;$ the wavenumber $k \in \mathbb{K} = k_c\mathbb{Z}$.

A general streamfunction

$$\psi = \sum_{k\in\mathbb{K}} \sum_{n\in\mathbb{N}^*} A(k,n)\,\psi_1(k,n) = \sum_{\mathbf{q}} A(\mathbf{q})\,\psi_1(\mathbf{q}) \quad \text{with} \quad \mathbf{q} = (k,n) \in \mathbb{K}\times\mathbb{N}^* .$$

To calculate the amplitudes $A(\mathbf{q})$... use the **adjoint eigenmodes**, the solutions of the **adjoint eigenproblem**

$$\sigma^* D^\dagger \cdot \phi = L_R^\dagger \cdot \phi$$

where the **adjoint operators** are defined by

$$\langle D\cdot\psi,\, \phi\rangle = \left\langle \psi,\, D^\dagger\cdot\phi \right\rangle \quad \text{and} \quad \langle L\cdot\psi,\, \phi\rangle = \left\langle \psi,\, L^\dagger\cdot\phi \right\rangle$$

where the inner product $\langle\psi,\,\phi\rangle = \displaystyle\int_{x=0}^{\lambda_c}\int_{z=-1}^{1} \psi(x,z)\,\phi^*(x,z)\,\frac{dx}{\lambda_c}\,\frac{dz}{2}$,

and one should care with the normalization of ϕ... $\langle D\cdot\psi_{1c},\, \phi_{1c}\rangle = 1$

$$\Longrightarrow \quad \text{the amplitude of the critical mode} \quad \boxed{A(\mathbf{q}_c) = \langle D\cdot\psi,\, \phi_{1c}\rangle} .$$

Weakly nonlinear analysis... requires the adjoint problem: ex 2.3

1 With the inner product $\langle \psi, \phi \rangle = \int_{x=0}^{\lambda_c} \int_{z=-1}^{1} \psi(x,z)\ \phi^*(x,z)\ \frac{dx}{\lambda_c}\ \frac{dz}{2}$,

one can define adjoint operators s.t.

$$\langle D \cdot \psi, \phi \rangle = \left\langle \psi, D^\dagger \cdot \phi \right\rangle \quad \text{and} \quad \langle L \cdot \psi, \phi \rangle = \left\langle \psi, L^\dagger \cdot \phi \right\rangle .$$

For Fourier modes in x, of wavenumber $k = mk_c$ with $m \in \mathbb{Z}^*$,

$$D = -\Delta = D^\dagger , \qquad L_R^\dagger \cdot \phi = -R^{-1}\Delta\Delta\phi - 2ikU'\partial_z\phi - ikU\Delta\phi .$$

2 Code the adjoint problem

$$\sigma^* D \cdot \phi = L_R^\dagger \cdot \phi$$

with the same spectral method as the one for the direct problem.

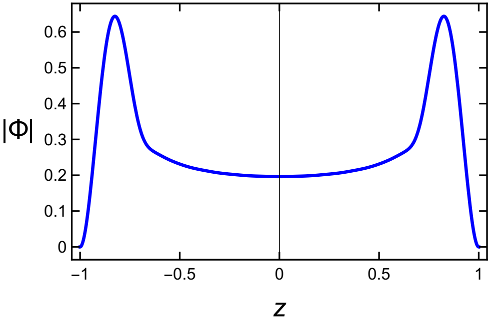
3.a Check: $k = k_c$, $R = R_c$ $\Rightarrow$ $\exists$ adjoint critical mode $\phi_{1c} = \Phi(z)\ \exp(ik_c x)$ corresponding to $\sigma = -i\omega_c$.

3.b Calculate $\Phi(z)$, plot $|\Phi(z)|$ and comment.

4 Normalize $\Phi(z)$ with the condition $\langle D \cdot \psi_{1c}, \phi_{1c} \rangle = 1$, $\psi_{1c} = \Psi(z)\ \exp(ik_c x)$ being the critical mode. Finally, replot $|\Phi(z)|$, and save the spectral coeff. of Φ to a file `U1`.

Weakly nonlinear analysis... requires the adjoint problem: ex 2.3

Plot of the modulus of the critical adjoint streamfunction $|\Phi(z)|$ vs z :



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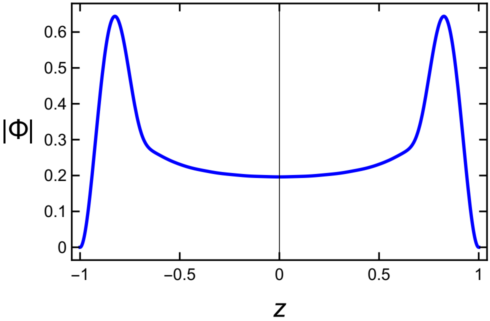

which describes the **receptivity** of the critical mode to **perturbations** !..

Weakly nonlinear analysis ?

- Near the bifurcation point, i.e., with the **bifurcation parameter**

$$\boxed{\epsilon = R/R_c - 1 \ll 1} \; .$$

- Uses the linear mode basis:
  modes $\psi_1(\mathbf{q})$ indexed with $\mathbf{q} = (k,n) = (x\text{-wavenumber}, z\text{-number})$

- Dominant modes are the critical ones $\mathbf{q} = \mathbf{q}_c = (k_c,1)$ or $\mathbf{q}_c^* = (-k_c,1)$,
  with eigenvalues

$$\sigma(\mathbf{q}_c,R) \;=\; -i\omega_c \;+\; (1+is)\epsilon/\tau_0 \;+\; O(\epsilon^2) \;, \quad \sigma(\mathbf{q}_c^*,R) \;=\; \sigma^*(\mathbf{q}_c,R) \;,$$

with $\tau_0 > 0$ the characteristic time, s the linear frequency-shift coefficient.

What do we know at the linear level ?

- With the **bifurcation parameter** $\boxed{\epsilon = R/R_c - 1 \ll 1}$.
- Dominant modes are the critical ones $\mathbf{q} = \mathbf{q}_c = (k_c,1)$ or $\mathbf{q}_c^* = (-k_c,1)$, with eigenvalues

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with $\tau_0 > 0$ the characteristic time, s the linear frequency-shift coefficient.

$\psi \;=\; A\;\exp(-i\omega_c t)\;\psi_{1c} \;+\; c.c.$ injected in

$$\boxed{D\cdot\partial_t\psi \;=\; L_R\cdot\psi} \quad\Longrightarrow$$

$$\left(\frac{dA}{dt} - i\omega_c A\right)\;\exp(-i\omega_c t)\;D\cdot\psi_{1c} \;+\; c.c. \quad=\quad A\;\exp(-i\omega_c t)\;L_R\cdot\psi_{1c} \;+\; c.c.$$

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$$=\; \sigma(\mathbf{q}_c, R)\; A\, \exp(-i\omega_c t)\; D \cdot \psi_{1c} \;+\; c.c.$$

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$$\left(\frac{dA}{dt} - i\omega_c A\right) \exp(-i\omega_c t) \ D \cdot \psi_{1c} \ + \ c.c. \ = \ A \ \exp(-i\omega_c t) \ L_R \cdot \psi_{1c} \ + \ c.c.$$

$$= \ \sigma(\mathbf{q}_c,R) \ A \ \exp(-i\omega_c t) \ D \cdot \psi_{1c} \ + \ c.c.$$

Projection onto the adjoint critical mode ϕ_{1c} $\Longrightarrow$

$$\frac{dA}{dt} - i\omega_c A \ = \ \sigma(\mathbf{q}_c,R) \ A \quad \Longleftrightarrow \quad \frac{dA}{dt} \ = \ [\sigma(\mathbf{q}_c,R) + i\omega_c] \ A \ \sim \ (1+is)\frac{\epsilon}{\tau_0} \ A \ ,$$

A is **exploding slowly** !

What do we know at the linear level ?

- With the **bifurcation parameter** $\boxed{\epsilon = R/R_c - 1 \ll 1}$.
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$$=\; \sigma(\mathbf{q}_c, R)\; A\; \exp(-i\omega_c t)\; D\cdot\psi_{1c} \;+\; c.c.$$

Projection onto the adjoint critical mode ϕ_{1c} $\Longrightarrow$

$$\frac{dA}{dt} - i\omega_c A \;=\; \sigma(\mathbf{q}_c, R)\; A \quad \Longleftrightarrow \quad \frac{dA}{dt} \;=\; [\sigma(\mathbf{q}_c, R) + i\omega_c]\; A \;\sim\; (1+is)\frac{\epsilon}{\tau_0}\; A \;,$$

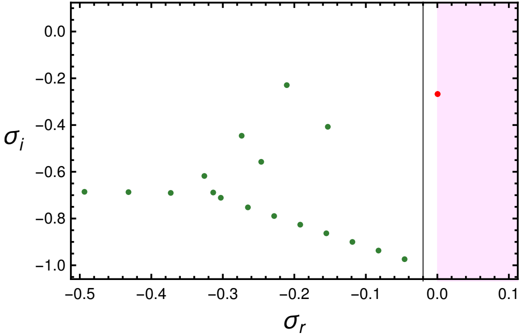
A is **exploding slowly** ! Must take into account **nonlinear effects** !

Weakly nonlinear analysis ?

- Near the bifurcation point, i.e., with the **bifurcation parameter**

$$\boxed{\epsilon = R/R_c - 1 \ll 1}\ .$$

- Correct the leading-order critical modes with passive modes, with the idea that '*long-living systems slave short-living systems*' (Haken), see the spectrum of the even modes of the linear problem at $k = k_c$, $\epsilon = 0.04$:



Weakly nonlinear analysis uses the linear mode basis

Modes $\psi_1(\mathbf{q})$ indexed with $\mathbf{q} = (k,n) = (x\text{-wavenumber},z\text{-number})$; $\epsilon = R/R_c - 1 \ll 1$

- **Active modes** correspond to $\mathbf{q} = \mathbf{q}_c = (k_c,1)$ or $\mathbf{q}_c^* = (-k_c,1)$ and have eigenvalues

$$\sigma(\mathbf{q}_c,R) \ = \ -i\omega_c \ + \ (1+is)\epsilon/\tau_0 \ + \ O(\epsilon^2) \ , \quad \sigma(\mathbf{q}_c^*,R) \ = \ \sigma^*(\mathbf{q}_c,R) \ .$$

- **Passive modes** correspond to $\mathbf{q} \neq \mathbf{q}_c,\mathbf{q}_c^*$ and are short-living (rapidly damped),

$$\sigma(\mathbf{q},R) \ = \ \sigma_r(\mathbf{q},R) \ + \ i\sigma_i(\mathbf{q},R) \quad \text{with} \quad \sigma_r(\mathbf{q},R) \ < \ \sigma_1 \ < \ 0 \ .$$

We seek an approximate solution of the nonlinear problem

$$\boxed{D \cdot \partial_t\psi \ = \ L_R \cdot \psi \ + \ N_2(\psi,\psi)} \qquad (*)$$

of the form

$$\psi \ = \ \psi_a \ + \ \psi_\perp \quad \text{with} \quad \psi_a \ = \ A \ \exp(-i\omega_c t) \ \psi_{1c} \ + \ c.c. \text{ the **active modes**}, \psi_a \ \ll \ 1 \ ,$$

$$\frac{dA}{dt} \ = \ O(\epsilon A)$$

and $\psi_\perp \ \ll \ \psi_a$ the **passive modes**.

Weakly nonlinear analysis: passive modes

are created by

$$N_2(\psi_a,\psi_a) \;=\; |A|^2 \,[\, \underbrace{N_2(\psi_{1c},\psi_{1c}^*) \;+\; c.c.}_{\text{mode } 0} \,] + [A^2 \; \underbrace{\exp(-2i\omega_c t)\; N_2(\psi_{1c},\psi_{1c})}_{\text{mode } 2k_c} \;+\; \underbrace{c.c.}_{\text{mode } -2k_c} \;] \,.$$

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The 'harmonic' modes of x-wavenumber $\pm 2k_c$ indicate the creation of **small scales**:

$$\text{modes of wavelength } \lambda_c \quad \rightarrow \quad \text{modes of wavelength } \lambda_c/2 \,.$$

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Hereafter, these modes are **neglected** !

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The x-homogeneous mode (‘mode 0’) corresponds to a **mean-flow correction**:

$$\psi_\perp(z,t) \;\simeq\; A_0(t)\; \psi_{20}(z)$$

with

$$\frac{dA_0}{dt}\; D\cdot\psi_{20} \;=\; A_0\; L_R\cdot\psi_{20} \;+\; |A|^2\; [N_2(\psi_{1c},\psi_{1c}^*) \;+\; c.c.]$$

Weakly nonlinear analysis: passive modes

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$\Longrightarrow$ with **quasistatic elimination**,

$$A_0 \;=\; |A|^2 \,, \quad \text{i.e.} \quad \psi_\perp \;\simeq\; |A|^2\; \psi_{20} \quad \text{and} \quad 0 \;=\; L_R\cdot\psi_{20} \;+\; [N_2(\psi_{1c},\psi_{1c}^*) \;+\; c.c.] \,.$$

Weakly nonlinear analysis: homogeneous passive mode: ex 2.4

$$\psi_\perp \;\simeq\; |A|^2\,\psi_{20} \quad \text{with} \quad 0 \;=\; L_R \cdot \psi_{20} \;+\; [N_2(\psi_{1c},\psi_{1c}^*) \;+\; c.c.]$$

To keep the **mean pressure gradient fixed** i.e. **fixed head losses**,
solve directly the x-component of the Navier-Stokes equation for $U_2 = -\psi_{20}'$,

$$R_c^{-1}\,U_2''(z) \;=\; [(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_1^* \;+\; c.c.]_x \qquad (\diamond)$$

with

$$\mathbf{u}_1 \;=\; -\,\partial_z[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_x \;+\; \partial_x[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_z \;.$$

Ex 2.4 : With Mathematica, writing formally

$$\Psi(z) \;=\; \Psi_r(z) \;+\; i\Psi_i(z)\;,$$

simplify the nonlinear term,

$$[(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_1^* \;+\; c.c.]_x \;=\; \qquad\qquad\qquad\qquad .$$

Weakly nonlinear analysis: homogeneous passive mode: ex 2.4

$$\psi_\perp \ \simeq\ |A|^2\ \psi_{20} \quad \text{with} \quad 0 \ =\ L_R \cdot \psi_{20} \ +\ [N_2(\psi_{1c},\psi_{1c}^*) \ +\ c.c.]$$

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Ex 2.4 : With Mathematica, writing formally

$$\Psi(z) \ =\ \Psi_r(z) \ +\ i\Psi_i(z) \ ,$$

simplify the nonlinear term,

$$[(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_1^* \ +\ c.c.]_x \ =\ 2k_c[\Psi_r''(z)\Psi_i(z) - \Psi_r(z)\Psi_i''(z)] \ .$$

To have $U_2 \neq 0$, it must be that both Ψ_r and $\Psi_i(z)$ do not vanish, see q 7.a of pb 2.1 !

Weakly nonlinear analysis: homogeneous passive mode: ex 2.5

$$\psi_\perp \simeq |A|^2 \psi_{20} \quad \text{with} \quad 0 = L_R \cdot \psi_{20} + [N_2(\psi_{1c},\psi_{1c}^*) + c.c.]$$

To keep the **mean pressure gradient fixed** i.e. **fixed head losses**,
solve directly the x-component of the Navier-Stokes equation

$$-D \cdot U_2 = \Delta U_2 = U_2''(z) = 2R_c k_c[\Psi_r''(z)\Psi_i(z) - \Psi_r(z)\Psi_i''(z)] \qquad (\diamond)$$

for $U_2(z)$ which satisfies the same BC as $\Psi(z)$:

$$U_2 = U_2' = 0 \quad \text{if} \quad z = \pm 1 \, .$$

1 Extract a part of the code of **pb 2.1** to compute the matrix MD that represents D for $k =$, with the spectral method.

2 Get the files written at **pb 2.1** to define the critical parameters R_c, k_c, then the real and imaginary parts of the critical streamfunction $\Psi(z)$. Evaluate the source term, the r.h.s. of $(\diamond)$, at the collocation points, to compute the source vector S_0 such that

$$-MD \cdot V_0 = S_0 \quad \text{with} \quad V_0 \text{ the vector of the spectral coefficients of } U_2(z) \, .$$

3 Solve this with the command `LinearSolve`, to compute V_0, then $U_2(z)$.
Plot $U_2(z)$ and explain the physics behind.

4 Save the vector V_0 to a file `U2`.

Weakly nonlinear analysis: homogeneous passive mode: ex 2.5

$$\psi_\perp \;\simeq\; |A|^2\;\psi_{20} \quad \text{with} \quad 0 \;=\; L_R\cdot\psi_{20} \;+\; [N_2(\psi_{1c},\psi_{1c}^*) \;+\; c.c.]$$

To keep the **mean pressure gradient fixed** i.e. **fixed head losses**,
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$$-D\cdot U_2 \;=\; \Delta U_2 \;=\; U_2''(z) \;=\; 2R_c k_c[\Psi_r''(z)\Psi_i(z) - \Psi_r(z)\Psi_i''(z)] \qquad (\diamond)$$

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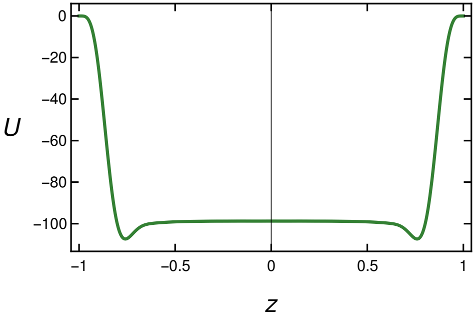
Weakly nonlinear analysis of PPF: homogeneous passive mode

$$\psi_\perp \;\simeq\; |A|^2 \, \psi_{20} \quad \text{with} \quad 0 \;=\; L_R \cdot \psi_{20} \;+\; [N_2(\psi_{1c}, \psi_{1c}^*) \;+\; c.c.]$$

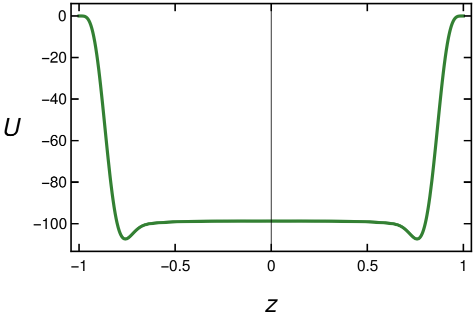
Plot of the correction to the basic flow $U_2(z) = -\psi'_{20}(z)$ vs z :



Weakly nonlinear analysis of PPF: homogeneous passive mode

$$\psi_\perp \simeq |A|^2 \psi_{20} \quad \text{with} \quad 0 = L_R \cdot \psi_{20} + [N_2(\psi_{1c},\psi_{1c}^*) + c.c.]$$

Plot of the correction to the basic flow $U_2(z) = -\psi_{20}'(z)$ vs z :



$\longleftrightarrow$ **reduction of the flow rate due to the transition !**

Weakly nonlinear analysis of PPF: feedback at order A^3

$$\psi \;=\; \psi_a + \psi_\perp \;, \quad \psi_a \;=\; A\, \exp(-i\omega_c t)\, \psi_{1c} \;+\; c.c. \;, \quad \psi_\perp \;\simeq\; |A|^2\, \psi_{20}$$

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi \;+\; N_2(\psi,\psi)} \qquad (*)$$

Weakly nonlinear analysis of PPF: feedback at order A^3

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Projection of

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi \;+\; N_2(\psi,\psi)} \qquad (*)$$

onto the adjoint critical mode ϕ_{1c}

$$\Longrightarrow \qquad \frac{dA}{dt} \;=\; (1+is)\frac{\epsilon}{\tau_0}\;A \;+\; \langle N_2(\psi,\psi),\,\phi_{1c}\rangle$$

Weakly nonlinear analysis of PPF: feedback at order A^3

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Projection of

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onto the adjoint critical mode ϕ_{1c}

$$\Longrightarrow \qquad \frac{dA}{dt} \ = \ (1+is)\frac{\epsilon}{\tau_0} \ A \ + \ \langle N_2(\psi,\psi), \ \phi_{1c} \rangle$$

Resonant terms:

$$\langle N_2(\psi,\psi), \ \phi_{1c} \rangle \ = \ g \ |A|^2 A$$

with the **feedback coefficient**

$$g \ = \ \langle N_2(\psi_{1c},\psi_{20}) + N_2(\psi_{20},\psi_{1c}), \ \phi_{1c} \rangle$$

that can be computed !

Weakly nonlinear analysis of PPF: feedback at order A^3 : ex 2.6

If one rewrites the nonlinear term in the vorticity equation as

$$\widetilde{N}_2(\mathbf{u}_a,\, \mathbf{u}_b) \;=\; \partial_x\big(\mathbf{u}_a \cdot \boldsymbol{\nabla} u_{zb}\big) - \partial_z\big(\mathbf{u}_a \cdot \boldsymbol{\nabla} u_{xb}\big) \;,$$

then the **nonlinear resonant term**

$$S_2(x,z) \;=\; \widetilde{N}_2(\mathbf{u}_1,\, U_2\mathbf{e}_x) \;+\; \widetilde{N}_2(U_2\mathbf{e}_x,\, \mathbf{u}_1) \;,$$

with

$$\mathbf{u}_1 \;=\; -\,\partial_z[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_x \;+\; \partial_x[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_z \;,$$

and the feedback coefficient

$$g \;=\; \langle S_2(x,z),\, \phi_{1c}(x,z)\rangle \;=\; \int_{z=-1}^{1} S_2(0,z)\; \Phi^*(z)\; \frac{dz}{2} \;.$$

Compute it with Mathematica, using the `NIntegrate` command;
show that

$$g = g_r + i g_i \quad \text{with} \quad g_r \;\text{ of a definite sign,} \qquad\qquad .$$

Weakly nonlinear analysis of PPF: feedback at order A^3 : ex 2.6

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$$S_2(x,z) \;=\; \widetilde{N}_2(\mathbf{u}_1,\, U_2\mathbf{e}_x) \;+\; \widetilde{N}_2(U_2\mathbf{e}_x,\, \mathbf{u}_1) \;,$$

with

$$\mathbf{u}_1 \;=\; -\,\partial_z[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_x \;+\; \partial_x[\Psi(z)\;\exp(ik_c x)]\;\mathbf{e}_z \;,$$

and the feedback coefficient

$$g \;=\; \langle S_2(x,z),\, \phi_{1c}(x,z)\rangle \;=\; \int_{z=-1}^{1} S_2(0,z)\;\Phi^*(z)\;\frac{dz}{2}\;.$$

Compute it with Mathematica, using the `NIntegrate` command;
show that

$$g = g_r + i g_i \quad \text{with} \quad g_r \;\text{ of a definite sign, }\; g_r \;>\; 0\;.$$

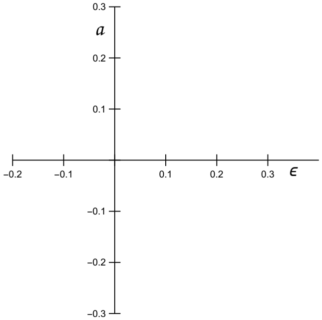
Weakly nonlinear analysis of PPF: feedback at order A^3

$$\frac{dA}{dt} \;=\; (1+is)\frac{\epsilon}{\tau_0}\; A + (g_r + ig_i)|A|^2 A$$

With a polar representation of the amplitude, $A \;=\; |A|\; \exp(i\phi)$,
the modulus $a = |A|$ satisfies the **amplitude equation**

$$\boxed{\frac{da}{dt} \;=\; \frac{\epsilon}{\tau_0}\; a + g_3 a^3} \quad \text{with} \quad g_3 \;=\; g_r \;>\; 0\;.$$

The fixed points and their stability properties may be determined...



Weakly nonlinear analysis of PPF: feedback at order A^3

$$\frac{dA}{dt} = (1+is)\frac{\epsilon}{\tau_0} A + (g_r + ig_i)|A|^2 A$$

With a polar representation of the amplitude, $A = |A| \exp(i\phi)$,
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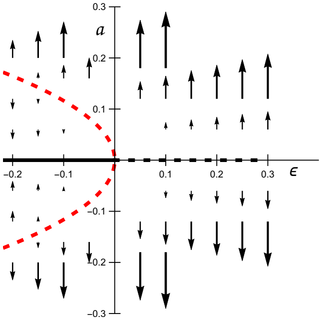
$$\boxed{\frac{da}{dt} = \frac{\epsilon}{\tau_0} a + g_3 a^3} \quad \text{with} \quad g_3 = g_r > 0 \; .$$

Subcritical pitchfork bifurcation:



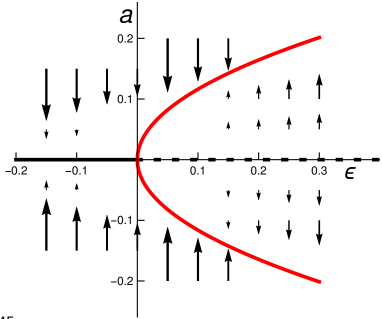
Weakly nonlinear analysis of RBT: feedback at order A^3

$$\frac{dA}{dt} = \frac{\epsilon}{\tau_0} A - g|A|^2 A$$

With a polar representation of the amplitude, $A = |A| \exp(i\phi)$,
the modulus $a = |A|$ satisfies the **amplitude equation**

$$\boxed{\frac{da}{dt} = \frac{\epsilon}{\tau_0} a - g\, a^3} \qquad \text{with} \qquad g > 0 \,.$$

Supercritical pitchfork bifurcation: **bifurcated solutions above onset,** $a \to 0$ as $\epsilon \to 0$:



Weakly nonlinear analysis of PPF: feedback at order A^3

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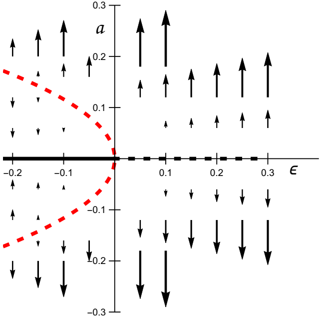
$$\boxed{\frac{da}{dt} = \frac{\epsilon}{\tau_0} a + g_3 a^3} \quad \text{with} \quad g_3 = g_r > 0 \;.$$

Subcritical pitchfork bifurcation: bifurcated solutions under onset, explosion above !?

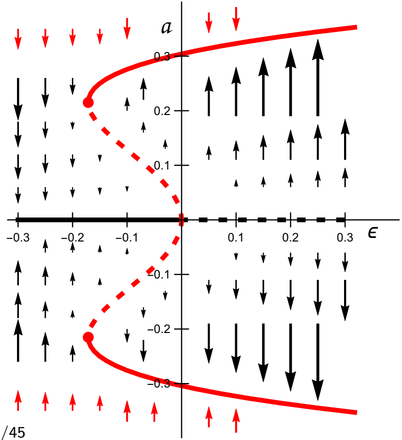


Weakly nonlinear analysis of PPF: subcritical bifurcation

More relevant model: add a saturation term at order A^5 or a^5 :

$$\boxed{\frac{da}{dt} = \frac{\epsilon}{\tau_0}\, a + g_3 a^3 - g_5 a^5} \quad \text{with} \quad g_3, g_5 > 0 \;,$$

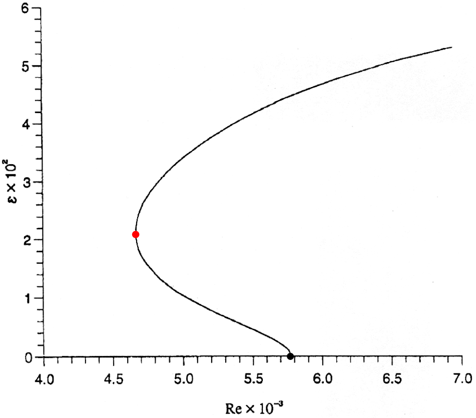
$\Longrightarrow$ **bistability** & **saddle-node bifurcations** - **quite 'abrupt' transitions**...



• : **turning points**

Nonlinear analysis of PPF: subcritical bifurcation

confirmed by strongly nonlinear computations:

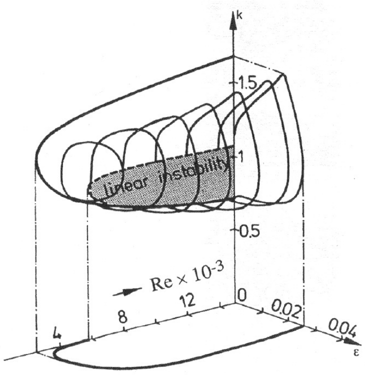


[Ehrenstein in Huerre & Rossi 1998 Hydrodynamics and NL instabilities. CUP]

Saddle-node bifurcation at the **turning point** • : waves appear from nowhere !

Nonlinear analysis of PPF: subcritical bifurcation

confirmed by strongly nonlinear computations:

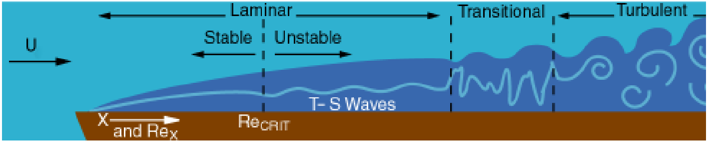


[Ehrenstein & Koch 1991 J. Fluid Mech.; Bayly *et al.* 1988 Ann. Rev. Fluid Mech.]

Transition can occur as soon as $R \geq 2900 \ll R_c = 5772$!

Globally subcritical scenarios of transition...

In the case of a boundary layer, e.g., the Blasius boundary layer, the flow is non-parallel and the transition develops in space



[Homsy *et al.* 2004 *Multimedia Fluid Mechanics*]

This may be studied with a **local spatial linear stability analysis**: compute modes in

$$\Psi(z)\ \exp[i(kx-\omega t)] \qquad \text{with} \qquad \omega\in\mathbb{R} \text{ the angular frequency, } \quad k=k(\omega,R,n)\in\mathbb{C} \text{ the spatial eigenvalues.}$$

Since $\exp(ikx) = \exp(ik_r x - k_i x)$, modes with $k_i < 0$ are amplified downstream...

$\hookrightarrow$ Phenomenological criterion to estimate the location where the flow becomes turbulent considering spatial amplification factors: 'e^N method'...

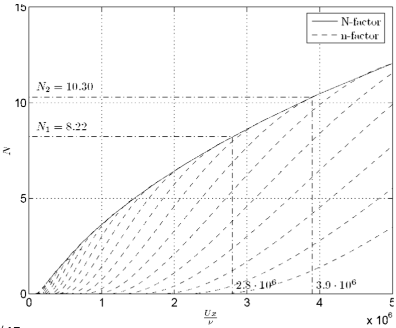
e^N method to predict transition in boundary layers

Between x and $x + dx$, the amplitude of the TS wave of angular frequency ω increases by

$$\frac{A+dA}{A} = e^{-k_i(x,\omega)\ dx} \iff d\ln A = -k_i(x,\omega)\ dx \implies A(x) = A(x=0)\ e^{n(x,\omega)}$$

with the **'amplification factor'** of the TS wave $n(x,\omega) = \int_{x_0(\omega)}^{x} -k_i(x',\omega)\ dx'$.

Compute with **local spatial linear stability analyses** $n(x,\omega)$ for a range of frequencies
→ set of n-curves
→ envelope = **maximum amplification factor** $N(x) = \max_\omega n(x,\omega)$



[Van Ingen 2008]

e^N method to predict transition in boundary layers

Ideas: 'linear' perturbations grow with an amplitude

$$A(x) \simeq A_0 \ e^{N(x)} \ .$$

The 'inlet' or 'leading edge' value of A scales with a power law of the **freestream turbulence level** Tu,

$$A_0 \simeq A_0' \ Tu^a \ .$$

e^N method to predict transition in boundary layers

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The 'inlet' or 'leading edge' value of A scales with a power law of the **freestream turbulence level** Tu,

$$A_0 \simeq A_0' \; Tu^a \; .$$

Transition to turbulence occurs when

$$A(x) \gtrsim A_c \iff e^{N(x)} \gtrsim \frac{A_c}{A_0} \iff N(x) \gtrsim \ln A_c - \ln A_0 = \ln A_c - \ln A_0' - a \ln Tu$$

$$\boxed{N(x) \gtrsim -8.43 - 2.4 \ln Tu}$$

[Mack 1977 Transition and laminar instability. *NASA - CR - 153203*]

Relevant for **airfoils** at **high** *Re* number, cf.

[Sørensen & Zahle 2014 Airfoil prediction at high Reynolds numbers using CFD. *EFMC10*] !..

e^N method to predict transition in boundary layers

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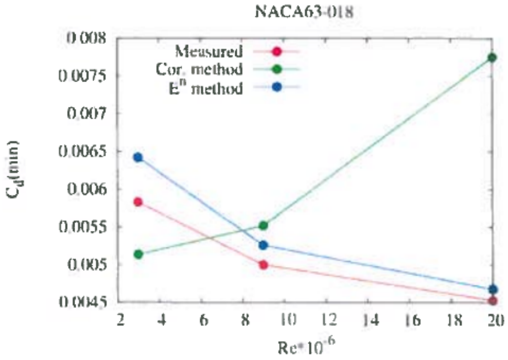
Relevant for **airfoils** at **high** *Re* number, cf. this curve of the minimum drag coeff. vs *Re* :

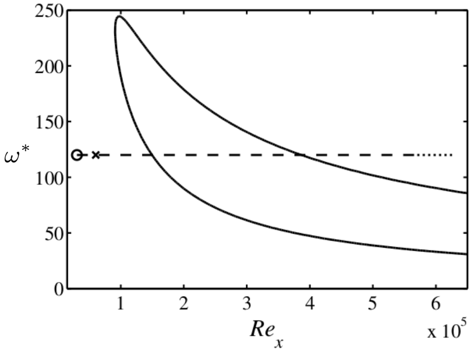


[Sørensen & Zahle 2014 Airfoil prediction at high Reynolds numbers using CFD. *EFMC10*]

see also [Bouville et al. 2018 Implementing the eN method into OpenFOAM. *SOpenFOAM WE*]

The local spatial stability analysis of a Blasius boundary layer does confirm the possible amplification of TS waves

According to Schlatter *et al.* 2010, region of amplification in the (Re_x,ω^*) plane, with $Re_x = \dfrac{Ux}{\nu}$ and ω^* a dimensionless frequency:

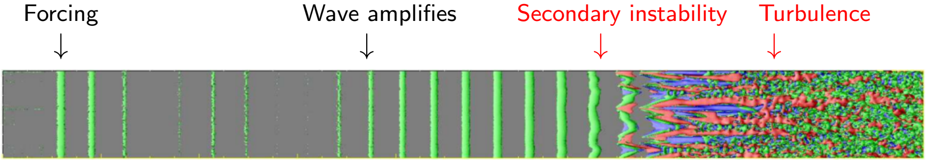


○ : inlet of the computational domain

× : position where the volume forcing is applied

Nonlinear simulations of a forced Blasius boundary layer, with 'large-eddy simulations', show a TS wave that goes to turbulence...

Case where the forcing consists
in an harmonic 2D force of temporal frequency ω^*
$+$ a small 3D noise, corresponding to a 'turbulence intensity' $\lesssim 0.1\%$:

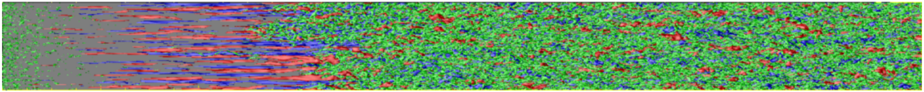


[Schlatter *et al.* 2010 *Int. J. Flow Control*]

Nonlinear simulations of a forced Blasius boundary layer, with 'large-eddy simulations', show a 'bypass' transition if the inflow is 'noisy', i.e., has a 'large' turbulence level

Case with the same forcing
but now the inflow or 'free-stream' has a 'turbulence intensity' $\simeq 5\%$:

Forcing ↓ Streamwise streaks ↓ Turbulence ↓



Abrupt transition typical of globally subcritical transition scenarios !