Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

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Session	Date	Content
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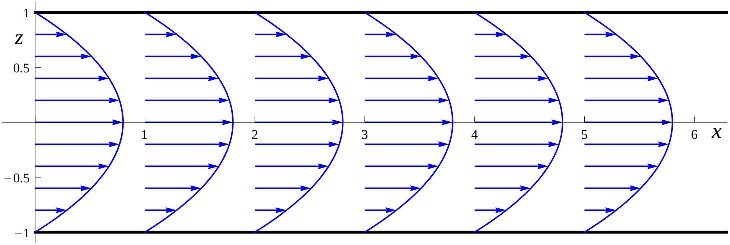
1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
2 -	06/10	**RB** Thermoconvection: linear stability analysis
3 -	13/10	**RB** Thermoconvection: (weakly) nonlinear phenomena
4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
→ 5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB* = Rayleigh-Bénard **OSF*** = Open Shear Flows

Today: session 5: transition in open shear flows:

- Numerical linear stability analysis of plane Poiseuille flow (PPF) : TS waves
- Launch HW3 !

When and how 2D xz laminar open shear flows get unstable ?
Example: plane Poiseuille flow



Viscous flow between two plates at $z = \pm h$:

$$\mathbf{v} \ = \ U(z)\,\mathbf{e}_x \ = \ U_0(1-(z/h)^2)\,\mathbf{e}_x \ , \quad p \ = \ p_{\text{static}} + \rho g Z \ = -Gx \quad \text{with} \quad G \ = \ 2\eta \frac{U_0}{h^2} \ .$$

Particular case of **plane parallel flow** !

When and how 2D xz laminar open shear flows get unstable ?
General example: plane parallel flows

$$\mathbf{v} = \mathbf{v}_0 = U(z)\,\mathbf{e}_x\ ,\quad p = p_{\text{static}} + \rho g Z = 0 \ \text{ in an inviscid fluid,}$$
$$p = p_{\text{static}} + \rho g Z = -Gx \ \text{ in a viscous fluid,}$$

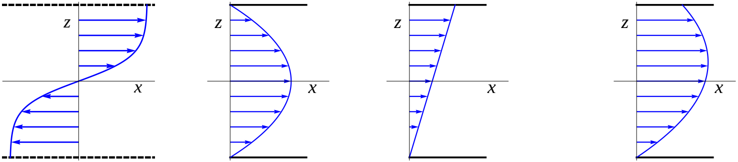
is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

$$\rho\big[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big] = -\boldsymbol{\nabla} p + \eta\Delta\mathbf{v}$$
$$\Longleftrightarrow \qquad \mathbf{0} = G\mathbf{e}_x + \eta U''(z)\mathbf{e}_x$$

whatever $U(z)$ in an inviscid fluid,

provided $U(z) = \alpha + \beta z + \gamma z^2$ in a viscous fluid.

Mixing layer | **Poiseuille flow** | **Couette flow** | **Couette-Poiseuille flow**



inviscid fl. | viscous fl. | viscous fl. | viscous fl.

Stability analysis of plane parallel flows

Basic flow:

$$\mathbf{v}_0 = U(z)\,\mathbf{e}_x\ , \quad p_0 = -Gx \quad \text{with} \quad \begin{array}{l} G = 0 \text{ in an inviscid fluid,} \\ G > 0 \text{ in a viscous fluid.} \end{array}$$

Basic flow with **perturbations**:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{u}\ , \quad p = p_0 + \widetilde{p}$$

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -(1/\rho)\boldsymbol{\nabla} p + \nu\Delta\mathbf{v} \qquad \text{(NS)}$$

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U\partial_x \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -(1/\rho)\boldsymbol{\nabla}\widetilde{p} + \nu\Delta\mathbf{u} \qquad \text{(NS)}$$

$$\mathrm{div}\mathbf{v} = \mathrm{div}\mathbf{u} = 0 \qquad \text{(MC)}$$

▷ Unit of length $= \ h$ half-width of the channel, thickness of the mixing layer...
▷ Unit of velocity $= \ U_0 = \max_z U(z)$ scale of U
▷ Unit of time $= \ h/U_0$ advection time

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U\partial_x \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\widetilde{p} + R^{-1}\,\Delta\mathbf{u} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R = U_0 h/\nu}$, $R = \infty$ in an inviscid fluid.

2D xz stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U \partial_x \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \widetilde{p} + R^{-1} \Delta \mathbf{u} , \qquad \text{(NS)}$$

$$\mathrm{div}\,\mathbf{u} = 0 . \qquad \text{(MC)}$$

2D xz **perturbations** can be defined by their **streamfunction** $\psi(x,z)$:

$$\mathbf{u} = \mathbf{curl}(\psi\, \mathbf{e}_y) = (\nabla\psi) \times \mathbf{e}_y = -(\partial_z \psi)\, \mathbf{e}_x + (\partial_x \psi)\, \mathbf{e}_z .$$

We eliminate $\widetilde{p}$ in (NS) by considering **curl**(NS) $\cdot\, \mathbf{e}_y$ i.e. the **vorticity equation**:

$$\partial_t(-\Delta\psi) + \left[\partial_z\left(\mathbf{u}\cdot\nabla u_x\right) - \partial_x\left(\mathbf{u}\cdot\nabla u_z\right)\right] = R^{-1}\Delta(-\Delta\psi) + U\partial_x(\Delta\psi) - U''(\partial_x\psi) \quad \text{(Vort)}$$

$$\Longleftrightarrow \quad \boxed{D \cdot \partial_t \psi = L_R \cdot \psi + N_2(\psi,\psi)} . \qquad \text{(Vort)}$$

2D xz stability analysis of plane parallel flows

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$$\Longleftrightarrow \quad \boxed{D \cdot \partial_t \psi \; = \; L_R \cdot \psi \; + \; N_2(\psi,\psi)} \; . \qquad \text{(Vort)}$$

Boundary conditions:

$$\text{viscous fluid :} \quad \mathbf{u} = \mathbf{0} \quad \Longleftrightarrow \quad \partial_x \psi \; = \; \partial_z \psi \; = \; 0 \quad \text{if} \quad z = z_\pm \; ,$$

$$\text{inviscid fluid :} \quad u_z = 0 \quad \Longleftrightarrow \quad \partial_x \psi \; = \; 0 \quad \text{if} \quad z = z_\pm \; .$$

2D xz linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \;=\; -\Delta \partial_t \psi \;, \quad L_R \cdot \psi \;=\; R^{-1} \Delta(-\Delta\psi) \;+\; U \partial_x (\Delta\psi) - U''(\partial_x \psi) \;,$$

viscous fluid : $\mathbf{u} = \mathbf{0} \iff \partial_x \psi \;=\; \partial_z \psi \;=\; 0$ if $z = z_\pm$,

inviscid fluid : $u_z = 0 \iff \partial_x \psi \;=\; 0$ if $z = z_\pm$.

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Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\;\exp(ikx + \sigma t)$$

with $k \;=$ **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
$\sigma \;=$ **temporal eigenvalue**.

2D xz linear stability analysis of plane parallel flows

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Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t) \;=\; \Psi_n(z)\; \exp[ik(x - c_r t)]\; \exp(k c_i t)$$

with k = **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
σ = **temporal eigenvalue**.

Most often the bulk velocity of the basic flow $\langle U \rangle_z > 0 \quad \Rightarrow$ by advection

$\sigma \;=\; -i\omega \;=\; -ikc$ with c the **complex phase velocity**,

$c_r > 0$ the **real phase velocity**,

$kc_i > 0$ (resp. < 0) the **growth rate** (resp. the opposite of the **damping rate**).

2D xz linear stability analysis of plane parallel flows

$$-\sigma\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; U\partial_x(\Delta\psi) - U''(\partial_x\psi) \qquad \text{(Vort)}$$

$$\iff \quad ikc\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; ikU\Delta\psi - ikU''\psi \qquad \text{(Vort)}$$

$$\iff \quad \boxed{(U-c)\Delta\psi \;-\; U''\psi \;=\; (ikR)^{-1}\Delta\Delta\psi} \qquad \text{(Vort)}$$

Orr - Sommerfeld eq. in a viscous fluid, **Rayleigh eq.** in an inviscid fluid ($R = \infty$)

BC at $z = z_\pm$: viscous fluid: $\psi \;=\; \partial_z\psi \;=\; 0$; inviscid fluid: $\psi \;=\; 0$.

Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t) \;=\; \Psi_n(z)\; \exp[ik(x - c_r t)]\; \exp(kc_i t)$$

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2D xz linear stability analysis of inviscid plane parallel flows

Normal mode analysis: assume ∃ one **amplified mode**

$$\psi = \Psi(z) \exp(ikx - ikct) = \Psi(z) \exp[ik(x - c_r t)] \exp(kc_i t)$$

with c_r the **real phase velocity**, $kc_i > 0$ the **growth rate**.

Satisfies **Rayleigh equation** $\boxed{(U - c)\Delta\psi - U''\psi = 0}$ with BC $\psi = 0$ at $z = z_\pm$.

Exercise 2.1 Rayleigh's inflection point criterion

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

▷ By multiplication with a suitable function and integration over $z \in [z_-, z_+]$, show that

$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz + \int_{z_-}^{z_+} \frac{U''(z)\,|\Psi(z)|^2}{U(z) - c} dz = 0$$

then $\displaystyle\int_{z_-}^{z_+} \frac{U''(z)\,|\Psi(z)|^2}{|U(z) - c|^2} dz = 0 \Rightarrow$ if $U'' \neq 0$, U'' must change sign somewhere,

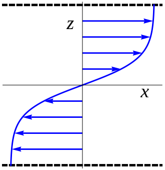
there must exist an **inflection point** in the U-profile.

▷ $U'' = 0$ everywhere $\Rightarrow$ contradiction $\Rightarrow$ **flow is stable (possibly only neutrally)**.

Instability of an inviscid plane parallel flow, the mixing layer

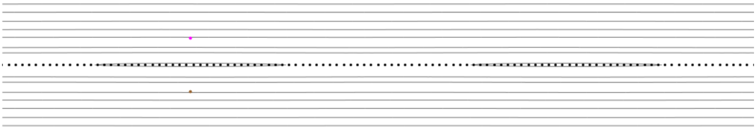
The hyperbolic tangent **mixing layer**

$$\mathbf{v}_0 = U_0 \tanh(z/h)\ \mathbf{e}_x$$



displays a **Kelvin-Helmholtz instability** !

Initial condition $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}$ with $\mathbf{u}$ small:



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Film sur la page web de ce module]

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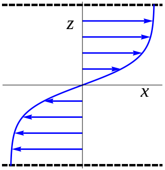

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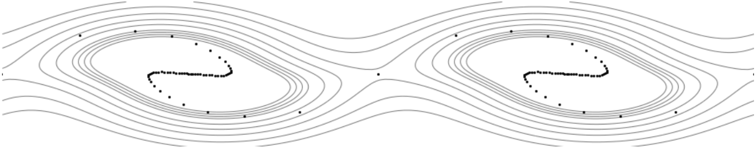
Time development: the perturbation **u** becomes large !



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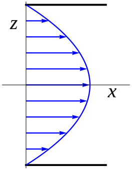
Stability of inviscid plane Poiseuille flow

Plane Poiseuille flow of an **inviscid fluid** has no inflection point $\Rightarrow$ it is **stable**.

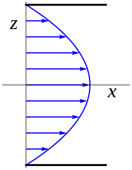
$$\mathbf{v}_0 = U_0(1-(z/h)^2)\ \mathbf{e}_x$$



Stability of viscous plane Poiseuille flow

Plane Poiseuille flow of a **viscous fluid** might be **unstable** ?

$$\mathbf{v}_0 = U_0(1-(z/h)^2)\ \mathbf{e}_x$$



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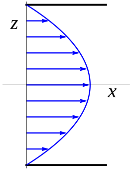


Must calculate normal modes

$$\psi = \Psi(z)\ \exp(ikx+\sigma t) = \Psi(z)\ \exp[ik(x-c_r t)]\ \exp(kc_i t)$$

by solving the **Orr - Sommerfeld equation**

$$\boxed{\sigma D\psi = -\sigma\Delta\psi = L_R\psi = -R^{-1}\Delta\Delta\psi + ik(U\Delta\psi - U''\psi)}$$

with the BC at $z=\pm 1$: $\psi = \partial_z\psi = 0.$

Eigenvalue $\sigma = -ikc$; $c_r = -\sigma_i/k$ phase velocity ;

$\sigma_r > 0 \leftrightarrow$ **amplified mode**
$\sigma_r = 0 \leftrightarrow$ **neutral mode**
$\sigma_r < 0 \leftrightarrow$ **damped mode**

Problem 2.1

Stability of viscous plane Poiseuille flow: pb. 2.1

$$\boxed{\sigma D\Psi \;=\; -\sigma\Delta\Psi \;=\; L_R\Psi \;=\; -R^{-1}\Delta\Delta\Psi \;+\; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(OS)}$$

$$\text{with} \quad \Delta \;=\; -k^2 \;+\; \frac{d^2}{dz^2}$$

and the boundary conditions $\quad \Psi \;=\; \Psi' \;=\; 0 \quad \text{if} \quad z = \pm 1$.

Spectral expansion taking into account the BC and even symmetry under $z \mapsto -z$:

$$\Psi(z) \;=\; \sum_{n=1}^{N} \Psi_n F_n(z)$$

$$\text{with} \quad F_n(z) \;=\; (z-1)^2\,(z+1)^2\;T_{2n-2}(z) \;=\; (z^2-1)^2\;T_{2n-2}(z)\;,$$

$$T_n(z) \;=\; n^{th} \text{ Chebyshev polynomial of the first kind.}$$

Evaluate (OS) at the **Gauss-Lobatto collocation points**

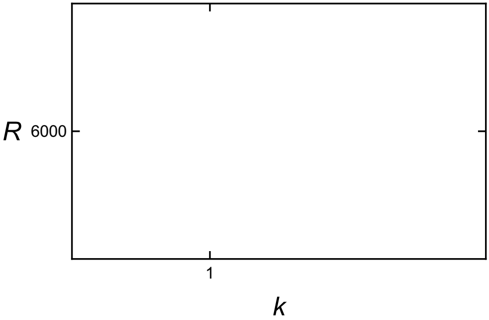
$$z_m \;=\; \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \{1,2,\cdots,N\}$$

$$\Longleftrightarrow \quad \sigma\sum_n \Psi_n DF_n(z_m) \;=\; \sum_n \Psi_n LF_n(z_m) \quad \Longleftrightarrow \quad \sigma MD\cdot V \;=\; ML\cdot V$$

$$\text{with} \quad V \;=\; (\Psi_1, \ldots, \Psi_N)^T \quad , \quad [MD]_{mn} \;=\; DF_n(z_m) \quad , \quad [ML]_{mn} \;=\; LF_n(z_m)\;.$$

Stability of viscous plane Poiseuille flow: pb. 2.1

Neutral curve:



converged, near the critical k corresponding to the minimal R, within 0.1% provided that

$$Nz \geq 17?, 18?, 19?$$