Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

http://emmanuelplaut.perso.univ-lorraine.fr/t2t

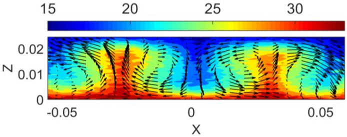
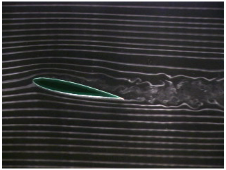
Session	Date	Content
1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
2 -	06/10	**RB** Thermoconvection: linear stability analysis
3 -	13/10	**RB** Thermoconvection: (weakly) nonlinear phenomena
→ 4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB* = Rayleigh-Bénard **OSF*** = Open Shear Flows

Today: session 4: transition in open shear flows:

- Introduction: OSF, instabilities of OSF, Rayleigh criterion
- Numerical linear stability analysis of plane Poiseuille flow: towards TS waves

Introduction: open shear flows, a new family of systems, quite different from Rayleigh-Bénard thermoconvection systems

	RB Thermoconvection	Open shear flows (OSF)
		
	[Leclerc & Métivier]	[Homsy et al.]
Fields	$\mathbf{v}$ T	$\mathbf{v}$ $T = constant$
Base state	$\mathbf{v} = \mathbf{0}$ trivial $T = T(z)$	$\mathbf{v} \neq \mathbf{0}$ complex $T = constant$
Eqs and nonlinearities	Navier-Stokes contains $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ Heat equation contains $\mathbf{v} \cdot \boldsymbol{\nabla} T$	Navier-Stokes contains $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ Heat equation trivially fulfilled

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	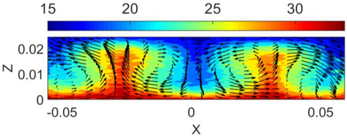	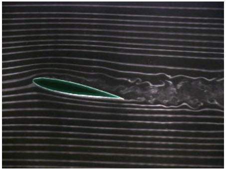
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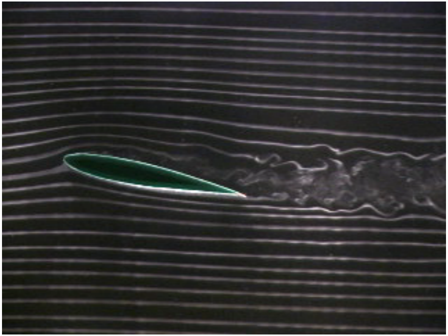
	Heat equation contains $\mathbf{v} \cdot \boldsymbol{\nabla} T$	Heat equation trivially fulfilled

OSF quite interesting but also quite challenging:

easier to understand $\mathbf{v} \cdot \boldsymbol{\nabla} T$ than $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$!

Open shear flows are often encountered in aerodynamics

Turbulent (?) **flow** around an obstacle, an airfoil, at an angle of attack $\alpha = 15^o$, observed with smoke in a wind tunnel at U. Stanford:



[Homsy et al. 2019 *Multimedia Fluid Mechanics Online*. Cambridge University Press
Films en bas de cette page web en fonction de α]

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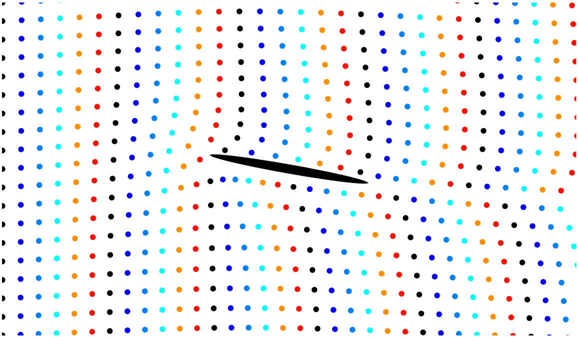
Laminar flow around an obstacle, an airfoil, also exists, and may be computed, for the external flow, with potential flow theory - complex analysis techniques:



[Plaut 2018 *Mécanique des fluides : des bases à la turbulence*. Cours Mines Nancy 2A. Film sur la page web de ce module]

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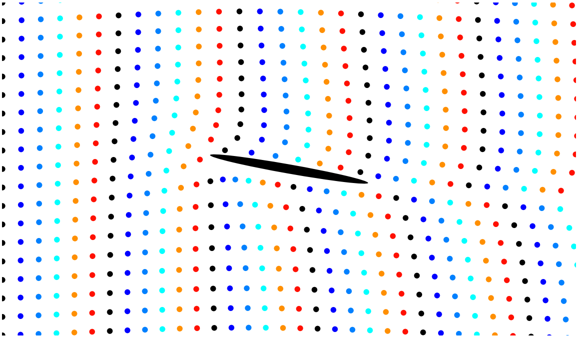

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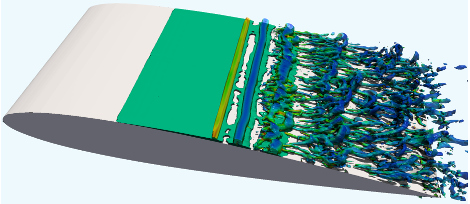
When and how **laminar open shear flows** get **unstable** and go to **turbulence ?**

When and how laminar open shear flows get unstable ?

In the case of **open shear flows** around an airfoil,
the **transition to turbulence develops into space**,
and it **changes the lift and drag !**

Hybrid Delayed Detached Eddy Simulations,
with the Spalart-Allmaras model, of **flow around an airfoil** at $Re = U_\infty c/\nu = 8\ 10^4$,
with a **laminar free stream** and an angle of attack $\alpha = 4^o$ that implies separation:



In green: iso-surface of zero streamwise velocity $\simeq$ separated region

In colors: Q iso-surfaces, coloured by vorticity magnitude $\simeq$ vortices

[Tangermann & Klein 2019 in *New Results in Numerical and Experimental Fluid Mechanics XII* - Springer
www.unibw.de/numerik]

Q criterion to detect numerically vortices

Vortices detected by the condition that **vorticity** dominates **strain** in the local gradient of velocity

$$\nabla \mathbf{v} \;=\; \Omega \;+\; \mathbf{S}$$

with the **rate-of-vorticity** tensor

$$\Omega \;=\; \frac{1}{2}\Big(\nabla \mathbf{v} \;-\; \nabla \mathbf{v}^T\Big)$$

and the **rate-of-strain** tensor

$$\mathbf{S} \;=\; \frac{1}{2}\Big(\nabla \mathbf{v} \;+\; \nabla \mathbf{v}^T\Big) \;.$$

Thus

$$Q \;=\; \frac{1}{2}\Big(\Omega : \Omega^T \;-\; \mathbf{S} : \mathbf{S}^T\Big) \;=\; \frac{1}{2}(\Omega_{ij}\Omega_{ij} - S_{ij}S_{ij}) \;=\; -\frac{1}{2}(\partial_{x_i} v_j)(\partial_{x_j} v_i) \;>\; 0 \;.$$

[Hunt, Wray & Moin 1988 Eddies, streams, and convergence zones in turbulent flows. *NASA report*; Jeong & Hussain 1995 On the identification of a vortex. *J. Fluid Mech.*]

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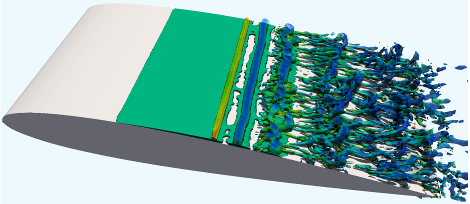

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This is **quite complex**

$\rightarrow$ we want to study **this question** in **simpler cases** !

When and how laminar open shear flows get unstable ?

When and how 2D xz laminar open shear flows get unstable ?

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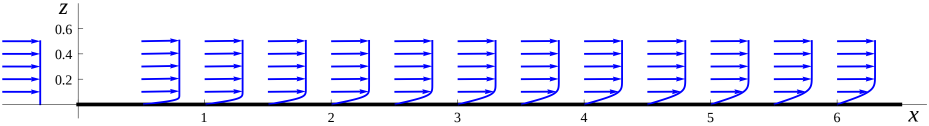
Example: Blasius boundary layer over a flat plate

Aerodynamical case: x and z in meters:



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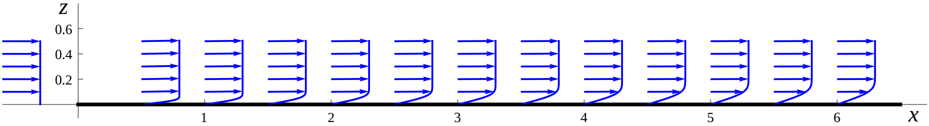

$$\delta = \sqrt{\frac{\nu x}{U}} , \quad \zeta = \frac{z}{\delta} , \quad v_x = U f'(\zeta) , \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\zeta f'(\zeta) - f(\zeta)]$$

Thickness of the boundary layer where $v_x = 0.99U$:

$$\delta_{99} = 5\sqrt{\frac{\nu x}{U}} \iff U = 25 \frac{\nu x}{\delta_{99}^2}$$

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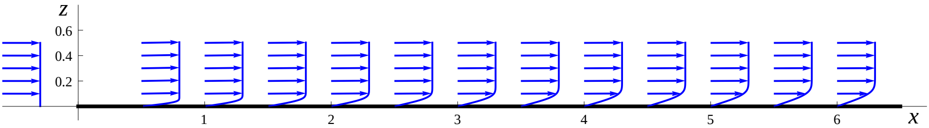

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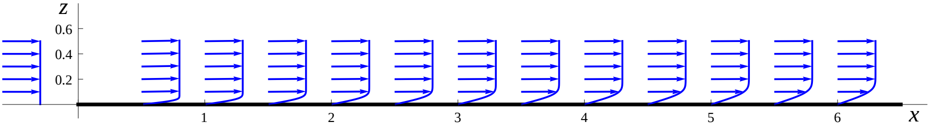

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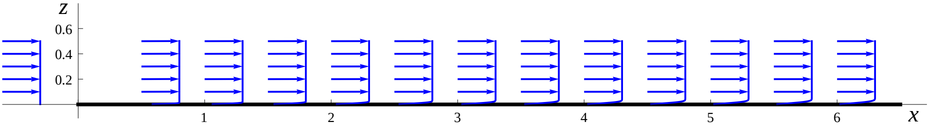
Aerodynamical case: x and z in meters: $U = 2$ m/s :



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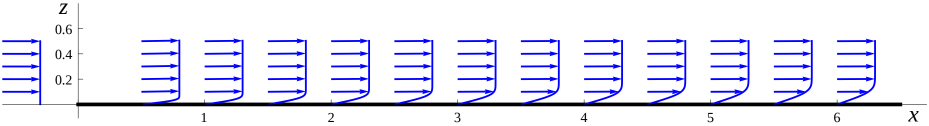

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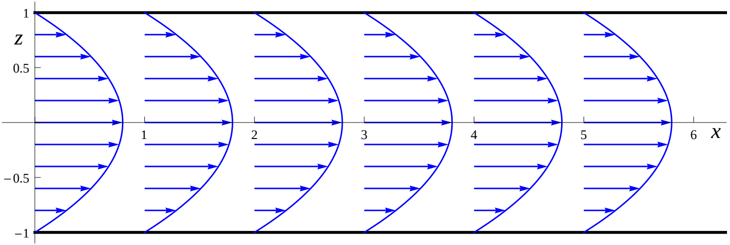

Viscous flow between two plates at $z = \pm h$: velocity and modified pressure:

$$\mathbf{v} \ = \ U(z)\ \mathbf{e}_x \ = \ U_0(1-(z/h)^2)\ \mathbf{e}_x \ , \quad p \ = \ p_{\text{static}} + \rho g Z \ =$$

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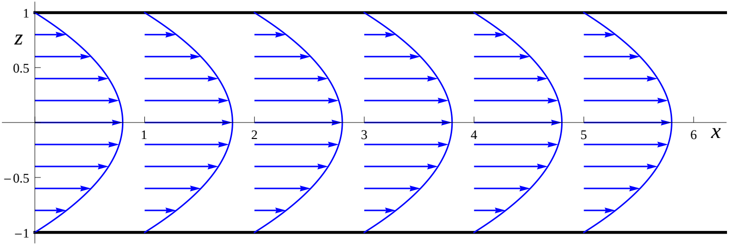

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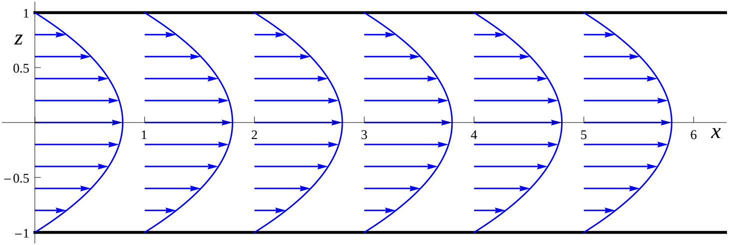

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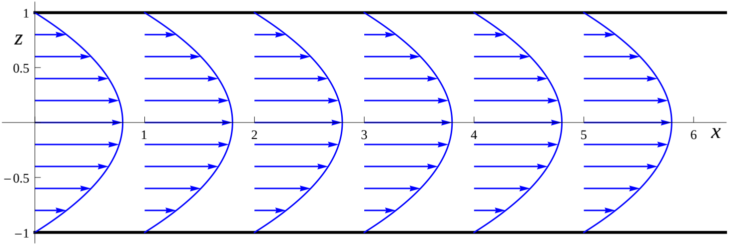

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Particular case of **plane parallel flow** !

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General example: plane parallel flows

$$\mathbf{v} = \mathbf{v}_0 = U(z)\,\mathbf{e}_x\ , \quad p = p_{\text{static}} + \rho g Z = 0 \text{ in an inviscid fluid,}$$
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is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

$$\rho\big[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big] = -\boldsymbol{\nabla} p + \eta\Delta\mathbf{v}$$
$$\iff \quad \mathbf{0} = G\mathbf{e}_x + \eta U''(z)\mathbf{e}_x$$

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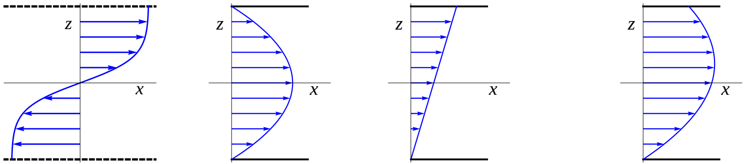
Mixing layer | **Poiseuille flow** | **Couette flow** | **Couette-Poiseuille flows**



inviscid fl. | viscous fl. | viscous fl. | viscous fl.

Stability analysis of plane parallel flows

Basic flow:

$$\mathbf{v}_0 = U(z)\,\mathbf{e}_x \;, \quad p_0 = -Gx \quad \text{with} \quad \begin{array}{l} G = 0 \text{ in an inviscid fluid,} \\ G > 0 \text{ in a viscous fluid.} \end{array}$$

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Basic flow with **perturbations**:

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$$\partial_t \mathbf{v} \;+\; (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \;=\; -(1/\rho)\boldsymbol{\nabla} p \;+\; \nu\Delta\mathbf{v} \qquad \text{(NS)}$$

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▷ Unit of time $=\;h/U_0$ advection time

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- ▷ Unit of length $= h$ half-width of the channel, thickness of the mixing layer...
- ▷ Unit of velocity $= U_0 = \max_z U(z)$ scale of U
- ▷ Unit of time $= h/U_0$ advection time

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U\partial_x \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\widetilde{p} + R^{-1}\,\Delta\mathbf{u} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R = U_0 h/\nu}$, $R = \infty$ in an inviscid fluid.

stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} \;+\; U' u_z \mathbf{e}_x \;+\; U \partial_x \mathbf{u} \;+\; (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -\boldsymbol{\nabla}\widetilde{p} \;+\; R^{-1}\,\Delta\mathbf{u}\;, \qquad \text{(NS)}$$

$$\mathrm{div}\mathbf{u} \;=\; 0\;. \qquad \text{(MC)}$$

2D *xz* stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} \ + \ U' u_z \mathbf{e}_x \ + \ U \partial_x \mathbf{u} \ + \ (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \ = \ -\boldsymbol{\nabla} \widetilde{p} \ + \ R^{-1} \ \Delta \mathbf{u} \ , \qquad \text{(NS)}$$

$$\mathrm{div} \mathbf{u} \ = \ 0 \ . \qquad \text{(MC)}$$

2D ***xz*** **perturbations** are most relevant, see **Tangermann's movies** or **Ex. 2.7** !

2D xz stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} + U' u_z \mathbf{e}_x + U \partial_x \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\widetilde{p} + R^{-1} \Delta \mathbf{u} , \qquad \text{(NS)}$$

$$\mathrm{div}\mathbf{u} = 0 . \qquad \text{(MC)}$$

2D xz perturbations are most relevant, see **Tangermann's movies** or **Ex. 2.7** !

They can be defined by their **streamfunction** $\psi(x,z)$:

$$\mathbf{u} = \mathbf{curl}(\psi \, \mathbf{e}_y) = (\boldsymbol{\nabla}\psi) \times \mathbf{e}_y = -(\partial_z \psi) \, \mathbf{e}_x + (\partial_x \psi) \, \mathbf{e}_z .$$

2D *xz* stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} \;+\; U' u_z \mathbf{e}_x \;+\; U\partial_x \mathbf{u} \;+\; (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -\boldsymbol{\nabla}\widetilde{p} \;+\; R^{-1}\,\Delta\mathbf{u}\;, \qquad \text{(NS)}$$

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How can one eliminate $\widetilde{p}$ in (NS) ?

2D *xz* stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

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How can one eliminate $\widetilde{p}$ in (NS) ? Consider **curl**(NS) $\cdot\,\mathbf{e}_y$ i.e. the **vorticity equation**:

$$\partial_t(-\Delta\psi) + \left[\partial_z\big(\mathbf{u}\cdot\boldsymbol{\nabla}u_x\big) - \partial_x\big(\mathbf{u}\cdot\boldsymbol{\nabla}u_z\big)\right] \;=\; R^{-1}\Delta(-\Delta\psi) + U\partial_x(\Delta\psi) - U''(\partial_x\psi)\;. \qquad \text{(Vort)}$$

2D xz stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

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$$\boxed{D\cdot\partial_t\psi \;=\; L_R\cdot\psi \;+\; N_2(\psi,\psi)}\;. \qquad \text{(Vort)}$$

2D *xz* stability analysis of plane parallel flows

Dimensionless equations for the **perturbations u** of velocity and $\widetilde{p}$ of pressure:

$$\partial_t \mathbf{u} \;+\; U' u_z \mathbf{e}_x \;+\; U\partial_x \mathbf{u} \;+\; (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -\boldsymbol{\nabla}\widetilde{p} \;+\; R^{-1}\,\Delta\mathbf{u}\;, \qquad \text{(NS)}$$

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2D *xz* perturbations are most relevant, see **Tangermann's movies** or **Ex. 2.7** !
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$$\boxed{D\cdot\partial_t\psi \;=\; L_R\cdot\psi \;+\; N_2(\psi,\psi)}\;. \qquad \text{(Vort)}$$

Boundary conditions:

viscous fluid : $\mathbf{u} = \mathbf{0} \iff \partial_x\psi = \partial_z\psi = 0$ if $z = z_\pm$,
inviscid fluid : $u_z = 0 \iff \partial_x\psi = 0$ if $z = z_\pm$.

2D xz linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \ = \ L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \ = \ -\Delta \partial_t \psi \ , \quad L_R \cdot \psi \ = \ R^{-1} \Delta(-\Delta\psi) \ + \ U \partial_x(\Delta\psi) - U''(\partial_x \psi) \ ,$$

$$\begin{array}{lllll} \text{viscous fluid :} & \mathbf{u} = \mathbf{0} & \iff & \partial_x \psi \ = \ \partial_z \psi \ = \ 0 & \text{if} \quad z = z_\pm \ , \\ \text{inviscid fluid :} & u_z = 0 & \iff & \partial_x \psi \ = \ 0 & \text{if} \quad z = z_\pm \ . \end{array}$$

2D xz linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi} \qquad \text{(Vort)}$$

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Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t)$$

with $k \;=\;$ **horizontal wavenumber**, $k \neq 0$, $\;n\;$ another label to mark normal modes,

2D xz linear stability analysis of plane parallel flows

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$\sigma \;=\;$ **temporal eigenvalue**.

2D xz linear stability analysis of plane parallel flows

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Normal mode analysis:

$$\psi \;=\; \Psi_n(z) \;\exp(ikx + \sigma t) \;=\; \Psi_n(z) \;\exp[ik(x - c_r t)] \;\exp(k c_i t)$$

with $k \;=\;$ **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
$\sigma \;=\;$ **temporal eigenvalue**.

Most often the bulk velocity of the basic flow $\langle U \rangle_z > 0 \quad \Rightarrow \quad$ by advection

$$\sigma \;=\; -i\omega \;=\; -ikc \quad \text{with} \quad c \text{ the } \textbf{complex phase velocity},$$

$c_r > 0$ the **real phase velocity**,

$kc_i > 0$ (resp. < 0) the **growth rate** (resp. the opposite of the **damping rate**).

2D xz linear stability analysis of plane parallel flows

$$-\sigma\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; U\partial_x(\Delta\psi) - U''(\partial_x\psi) \qquad \text{(Vort)}$$

BC at $z = z_\pm$: viscous fluid: $\psi \;=\; \partial_z\psi \;=\; 0$; inviscid fluid: $\psi \;=\; 0$.

Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\;\exp(ikx+\sigma t) \;=\; \Psi_n(z)\;\exp[ik(x-c_r t)]\;\exp(kc_i t)$$

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2D xz linear stability analysis of plane parallel flows

$$-\sigma\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; U\partial_x(\Delta\psi) - U''(\partial_x\psi) \qquad \text{(Vort)}$$

$$\Longleftrightarrow \quad ikc\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; ikU\Delta\psi - ikU''\psi \qquad \text{(Vort)}$$

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2D xz linear stability analysis of plane parallel flows

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$$\Longleftrightarrow \quad ikc\Delta\psi \;=\; R^{-1}\Delta(-\Delta\psi) \;+\; ikU\Delta\psi - ikU''\psi \qquad \text{(Vort)}$$

$$\Longleftrightarrow \quad \boxed{(U-c)\Delta\psi \;-\; U''\psi \;=\; (ikR)^{-1}\Delta\Delta\psi} \qquad \text{(Vort)}$$

Orr - Sommerfeld eq. in a viscous fluid, **Rayleigh eq.** in an inviscid fluid ($R = \infty$)

BC at $z = z_\pm$: viscous fluid: $\psi \;=\; \partial_z\psi \;=\; 0$; inviscid fluid: $\psi \;=\; 0$.

Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t) \;=\; \Psi_n(z)\; \exp[ik(x - c_r t)]\; \exp(kc_i t)$$

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2D xz linear stability analysis of inviscid plane parallel flows

Normal mode analysis: assume $\exists$ one **amplified mode**

$$\psi \;=\; \Psi(z)\; \exp(ikx - ikct) \;=\; \Psi(z)\; \exp[ik(x - c_r t)]\; \exp(kc_i t)$$

with c_r the **real phase velocity**, $kc_i > 0$ the **growth rate**.

Satisfies **Rayleigh equation** $\boxed{(U - c)\Delta\psi \;-\; U''\psi \;=\; 0}$ with BC $\psi \;=\; 0$ at $z = z_\pm$.

2D xz linear stability analysis of inviscid plane parallel flows

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$$\psi = \Psi(z) \exp(ikx - ikct) = \Psi(z) \exp[ik(x - c_r t)] \exp(kc_i t)$$

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Ex. 2.1

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

2D xz linear stability analysis of inviscid plane parallel flows

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Ex. 2.1

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

▷ By multiplication with a suitable function and integration over $z \in [z_-, z_+]$, show that

$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz + \int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{U(z) - c} dz = 0$$

then $$\int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{|U(z) - c|^2} dz = 0$$

2D *xz* linear stability analysis of inviscid plane parallel flows

Normal mode analysis: assume $\exists$ one **amplified mode**

$$\psi \;=\; \Psi(z)\;\exp(ikx - ikct) \;=\; \Psi(z)\;\exp[ik(x-c_r t)]\;\exp(kc_i t)$$

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Ex. 2.1 **Rayleigh's inflection point criterion**

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

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$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz \;+\; \int_{z_-}^{z_+} \frac{U''(z)\;|\Psi(z)|^2}{U(z)-c}\,dz \;=\; 0$$

then $\displaystyle\int_{z_-}^{z_+} \frac{U''(z)\;|\Psi(z)|^2}{|U(z)-c|^2}\,dz \;=\; 0 \;\Rightarrow\;$ if $U'' \neq 0$, U'' must change sign somewhere,
there must exist an **inflection point** in the *U*-profile.

2D xz linear stability analysis of inviscid plane parallel flows

Normal mode analysis: assume ∃ one **amplified mode**

$$\psi \;=\; \Psi(z)\;\exp(ikx - ikct) \;=\; \Psi(z)\;\exp[ik(x - c_r t)]\;\exp(kc_i t)$$

with c_r the **real phase velocity**, $kc_i > 0$ the **growth rate**.

Satisfies **Rayleigh equation** $\boxed{(U-c)\Delta\psi \;-\; U''\psi \;=\; 0}$ with BC $\psi \;=\; 0$ at $z = z_\pm$.

Ex. 2.1 **Rayleigh's inflection point criterion**

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

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$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz \;+\; \int_{z_-}^{z_+} \frac{U''(z)\;|\Psi(z)|^2}{U(z) - c} dz \;=\; 0$$

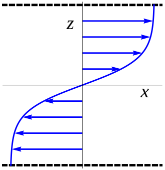
then $\displaystyle\int_{z_-}^{z_+} \frac{U''(z)\;|\Psi(z)|^2}{|U(z) - c|^2} dz \;=\; 0 \;\Rightarrow$ if $U'' \neq 0$, U'' must change sign somewhere,
there must exist an **inflection point** in the U-profile.

▷ $U'' = 0$ everywhere $\Rightarrow$ contradiction $\Rightarrow$ **flow is stable (possibly only neutrally)**.

Instability of an inviscid plane parallel flow, the mixing layer

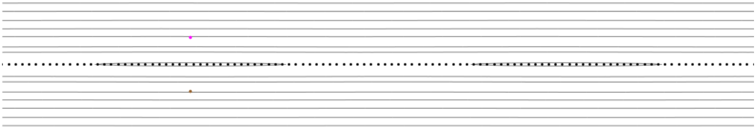
The hyperbolic tangent **mixing layer**

$$\mathbf{v}_0 \;=\; U_0 \tanh(z/h)\; \mathbf{e}_x$$



displays a **Kelvin-Helmholtz Instability** !

Initial condition $\mathbf{v} \;=\; \mathbf{v}_0 \,+\, \mathbf{u}$ with $\mathbf{u}$ small:



[Plaut 2018 *Mécanique des fluides : des bases à la turbulence*. Cours Mines Nancy 2A.
Film sur la page web de ce module]

Instability of an inviscid plane parallel flow, the mixing layer

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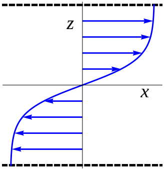
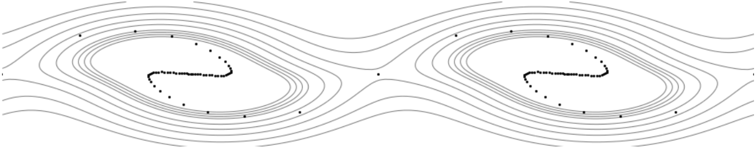


displays a **Kelvin-Helmholtz instability** !

Time development: the perturbation $\mathbf{u}$ becomes large !



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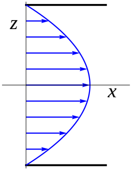
Stability of inviscid plane Poiseuille flow

Plane Poiseuille flow of an **inviscid fluid** has no inflection point $\Rightarrow$ it is **stable**.

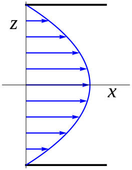
$$\mathbf{v}_0 = U_0(1 - (z/h)^2)\ \mathbf{e}_x$$



Stability of viscous plane Poiseuille flow

Plane Poiseuille flow of a **viscous fluid** might be **unstable** ?

$$\mathbf{v}_0 = (1 - z^2)\ \mathbf{e}_x$$



Stability of viscous plane Poiseuille flow

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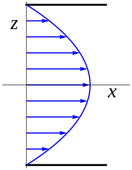
$$\mathbf{v}_0 \;=\; (1 - z^2)\;\mathbf{e}_x$$



Must calculate normal modes

$$\psi \;=\; \Psi(z)\;\exp(ikx + \sigma t) \;=\; \Psi(z)\;\exp[ik(x - c_r t)]\;\exp(kc_i t)$$

by solving the **Orr - Sommerfeld equation**

$$\boxed{\sigma D\psi \;=\; -\sigma\Delta\psi \;=\; L_R\psi \;=\; -R^{-1}\Delta\Delta\psi \;+\; ik(U\Delta\psi - U''\psi)}$$

with the BC at $z = \pm 1$: $\qquad \psi \;=\; \partial_z\psi \;=\; 0.$

Eigenvalue $\sigma = -ikc$; $c_r = -\sigma_i/k$ phase velocity ;
$\sigma_r > 0 \;\leftrightarrow\;$ **amplified mode**
$\sigma_r = 0 \;\leftrightarrow\;$ **neutral mode**
$\sigma_r < 0 \;\leftrightarrow\;$ **damped mode**

Problem 2.1

Stability of viscous plane Poiseuille flow: problem 2.1

$$\boxed{\sigma D\Psi \; = \; -\sigma\Delta\Psi \; = \; L_R\Psi \; = \; -R^{-1}\Delta\Delta\Psi \; + \; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(OS)}$$

$$\text{with} \quad \Delta \; = \; -k^2 \; + \; \frac{d^2}{dz^2}$$

and the boundary conditions $\quad \Psi \; = \; \Psi' \; = \; 0 \quad \text{if} \quad z = \pm 1$.

Stability of viscous plane Poiseuille flow: problem 2.1

$$\boxed{\sigma D\Psi \;=\; -\sigma\Delta\Psi \;=\; L_R\Psi \;=\; -R^{-1}\Delta\Delta\Psi \;+\; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(OS)}$$

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and the boundary conditions $\quad \Psi \;=\; \Psi' \;=\; 0 \quad \text{if} \quad z = \pm 1 \;.$

Spectral expansion taking into account the BC and even symmetry under $z \mapsto -z$:

$$\Psi(z) \;=\; \sum_{n=1}^{N} \Psi_n F_n(z)$$

$$\text{with} \quad F_n(z) \;=\; (z-1)^2\,(z+1)^2\;T_{2n-2}(z) \;=\; (z^2-1)^2\;T_{2n-2}(z)\;,$$

$$T_n(z) \;=\; n^{th} \text{ Chebyshev polynomial of the first kind.}$$

Stability of viscous plane Poiseuille flow: problem 2.1

$$\boxed{\sigma D\Psi \; = \; -\sigma\Delta\Psi \; = \; L_R\Psi \; = \; -R^{-1}\Delta\Delta\Psi \; + \; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(OS)}$$

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Evaluate (OS) at the **Gauss-Lobatto collocation points**

$$z_m \; = \; \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \; \{1,2,\cdots,N\}$$

$$\iff \quad \sigma\sum_n \Psi_n DF_n(z_m) \; = \; \sum_n \Psi_n LF_n(z_m) \quad \iff \quad \sigma MD\cdot V \; = \; ML\cdot V$$

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