Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

Session	Date	Content
1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
2 -	06/10	**RB** Thermoconvection: linear stability analysis
→ 3 -	13/10	**RB** Thermoconvection: (weakly) nonlinear phenomena
4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB = Rayleigh-Bénard **OSF** = Open Shear Flows

Give me homeworks 1 & 2 as defined in

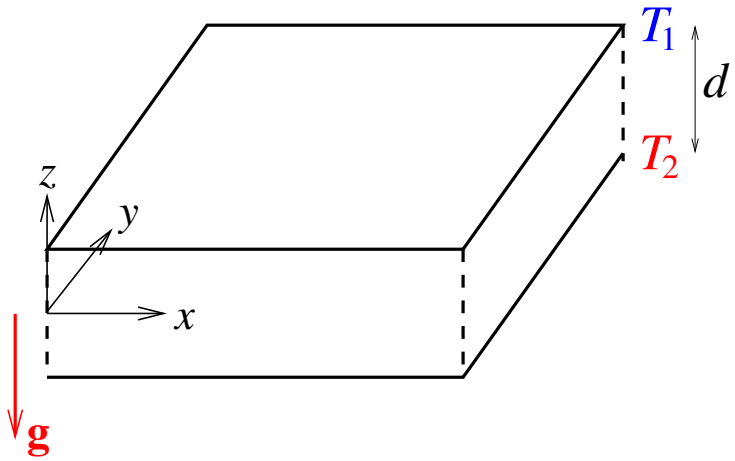
http://emmanuelplaut.perso.univ-lorraine.fr/t2t .

Transition to spatial complexity in Rayleigh-Bénard thermoconvection

1. The system: equations and (stress-free) slip boundary conditions
2. Linear stability analysis: time dependence - ex. 1.1, 1.2 & 1.3, $A(t) = ?$
   direct and adjoint mode bases... at order A...
3. Weakly nonlinear analysis: calculate an amplitude equation $\partial_t A = ?$
   - Quasistatic elimination of the passive mode at order A^2 - Nusselt number
   - Resonant terms at order A^3
   - Amplitude equation - Supercritical bifurcation
4. Results for other BC, another geometry, and in the nonlinear regime
   - No-slip BC: results in extended geometry
   - Slip BC: a glimpse at the Lorenz model and chaos
   - No-slip BC: results in confined geometry - Flow reversals...

Rayleigh-Bénard Thermoconvection: dimensionless model

- Unit of length = thickness d
- Unit of time = heat diffus° time $\tau_{\text{therm}} = \dfrac{d^2}{\kappa}$
- Unit of velocity = $V = \dfrac{d}{\tau_{\text{therm}}} = \dfrac{\kappa}{d}$
- Unit of temperature = $\delta T = T_2 - T_1$



Introduce a dimensionless perturbat° of temperature θ, s.t. the dimensionless temperature

$$T' = T'_0 - z' + \theta$$

$\Rightarrow$ dimensionless **Oberbeck - Boussinesq equations**

$$\text{div}\mathbf{v} = 0 , \qquad \text{(MC)}$$

$$P^{-1}\frac{d\mathbf{v}}{dt} = R\theta\, \mathbf{e}_z - \nabla p + \Delta \mathbf{v} , \qquad \text{(NS)}$$

$$\frac{d\theta}{dt} = \Delta\theta + v_z , \qquad \text{(HE)}$$

with the **Rayleigh number** $R = \alpha\, \delta T\, g\, d^3/(\kappa\nu)$ and the **Prandtl number** $P = \nu/\kappa$.

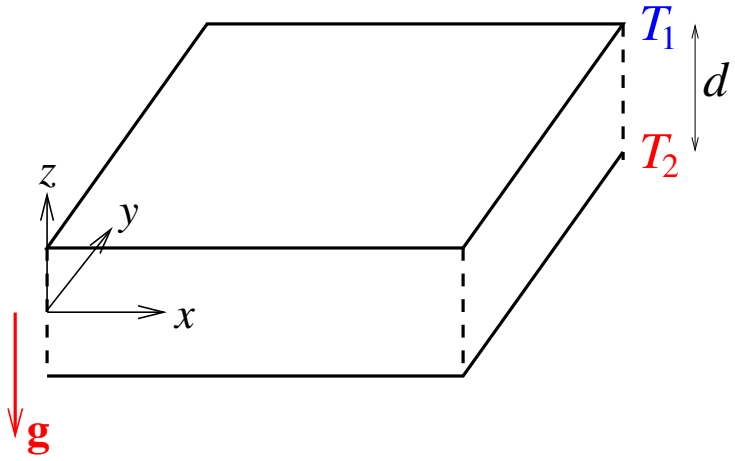
Rayleigh-Bénard Thermoconvection: dimensionless model

OB equations:

$$\mathrm{div}\mathbf{v} \;=\; 0\;, \qquad \text{(MC)}$$

$$P^{-1}\frac{d\mathbf{v}}{dt} \;=\; R\theta\,\mathbf{e}_z \;-\; \boldsymbol{\nabla} p \;+\; \Delta\mathbf{v}\;, \qquad \text{(NS)}$$

$$\frac{d\theta}{dt} \;=\; \Delta\theta \;+\; v_z\;, \qquad \text{(HE)}$$



Isotropy of the problem in the horizontal plane $\Rightarrow$ focus on **2D** xz **solutions**

$$\mathbf{v} \;=\; v_x(x,z,t)\,\mathbf{e}_x \;+\; v_z(x,z,t)\,\mathbf{e}_z\;, \quad \theta \;=\; \theta(x,z,t)\;.$$

Solve (MC) by introducing a **streamfunction** ψ such that

$$\mathbf{v} \;=\; \mathbf{curl}(\psi\,\mathbf{e}_y) \;=\; (\boldsymbol{\nabla}\psi)\times\mathbf{e}_y \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z\;.$$

Eliminate p in (NS) by considering $\mathbf{curl}(\text{NS})\cdot\mathbf{e}_y$ i.e. the **vorticity equation**:

$$P^{-1}\partial_t(-\Delta\psi) \;+\; P^{-1}\big[\partial_z(\mathbf{v}\cdot\boldsymbol{\nabla}v_x) - \partial_x(\mathbf{v}\cdot\boldsymbol{\nabla}v_z)\big] \;=\; -R\partial_x\theta \;+\; \Delta(-\Delta\psi)\;. \qquad \text{(Vort)}$$

RBT: 2D xz model with slip BC

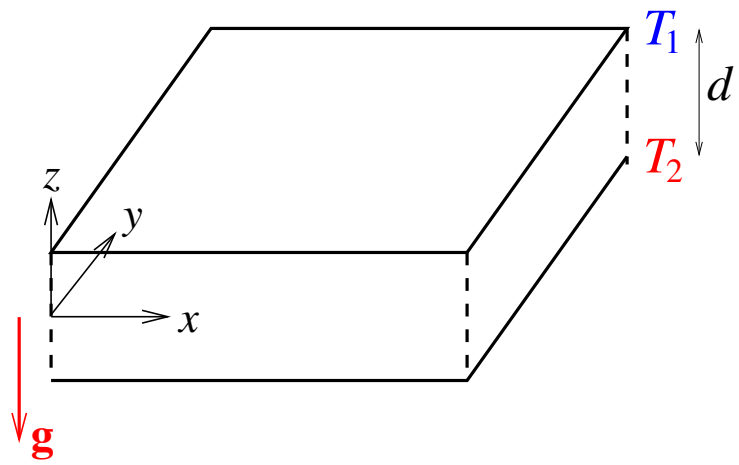
Local state vector: $V = (\psi,\theta)$ s. t.

$$\mathbf{v} = -(\partial_z\psi)\,\mathbf{e}_x + (\partial_x\psi)\,\mathbf{e}_z\ ,$$

$$T = T_0 - z + \theta\ ,$$

obeys the system of coupled PDE

$$\boxed{D\cdot\partial_t V = L_R\cdot V + N_2(V,V)}\ .$$



$$[D\cdot\partial_t V]_\psi = P^{-1}(-\Delta\partial_t\psi)\ ,\quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi)\ , \qquad \text{(Vort)}$$

$$[N_2(V,V)]_\psi = P^{-1}\big[\partial_x(\mathbf{v}\cdot\boldsymbol{\nabla}v_z) - \partial_z(\mathbf{v}\cdot\boldsymbol{\nabla}v_x)\big]\ , \qquad \text{(Vort)}$$

$$[D\cdot\partial_t V]_\theta = \partial_t\theta\ ,\quad [L_R\cdot V]_\theta = \Delta\theta + v_z\ ,\quad [N_2(V,V)]_\theta = -\mathbf{v}\cdot\boldsymbol{\nabla}\theta\ . \qquad \text{(HE)}$$

Boundary conditions on θ : **isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

Boundary conditions on ψ i.e. $\mathbf{v}$: **slip without stress ('stress-free'):**

$$v_z = 0 \quad\text{and}\quad \tau_{xz} = \partial_z v_x = 0 \iff \partial_x\psi = \partial_z^2\psi = 0 \quad\text{if}\quad z = \pm 1/2\ .$$

Extended geometry in the xy plane: (no BC or) periodic BC under $x \mapsto x + L$.

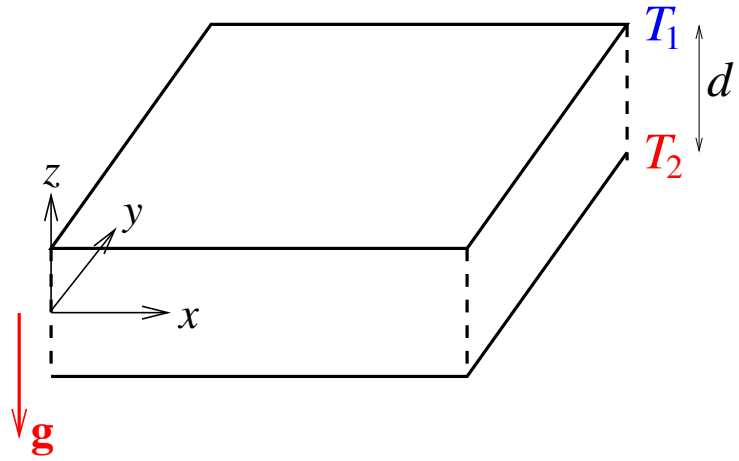
RBT: 2D xz model with slip BC: linear stability analysis

Local state vector: $V = (\psi,\theta)$ s. t.

$$\mathbf{v} = -(\partial_z\psi)\,\mathbf{e}_x + (\partial_x\psi)\,\mathbf{e}_z\,,$$

$$T = T_0 - z + \theta\,,$$

Eigenproblem: $\boxed{\sigma\, D\cdot V = L_R\cdot V}$.



Ex. 1.1 and 1.2: normal mode analysis: the solution of the initial value problem is the superposition of **normal modes** that are Fourier modes in $\exp(ikx)$,

$$V = V_1(k,\pm,n) = (\Psi,\,\Theta)\,\exp(ikx)\,\sin(n\pi z + n\pi/2)$$

$$k = \textbf{horizontal wavenumber},\ k\neq 0,$$

$$n \leftrightarrow \text{dependence on } z\ ,\ \pm \leftrightarrow 2 \text{ modes at fixed } k \text{ and } n,$$

$$\sigma = \sigma(k,\pm,n,R,P) = \textbf{temporal eigenvalue}$$

Most relevant normal modes $\leftrightarrow (\pm,n) = (+,1)$: $V = (\Psi,\,\Theta)\,\exp(ikx)\,\cos(\pi z)$

$$\text{(HE)} \implies \Psi = -\frac{i}{k}(D_1+\sigma)\,\Theta \quad \text{with} \quad D_1 = -\Delta = k^2+\pi^2$$

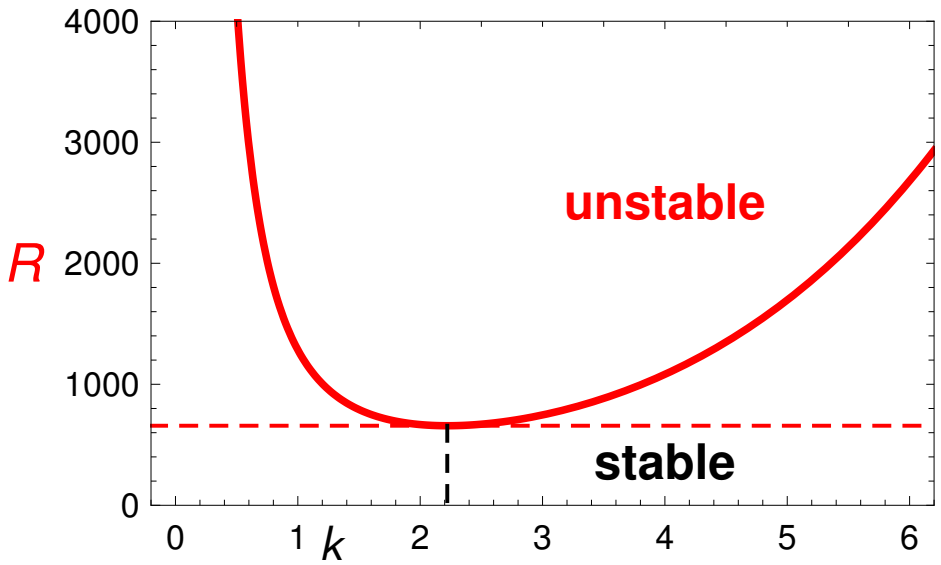
$$\text{(Vort)} \implies \sigma^2 + (1+P)D_1\sigma + P(D_1^3 - Rk^2)/D_1 = 0$$

RBT: 2D xz model with slip BC: linear stability analysis

This characteristic equation for the **temporal eigenvalue** has 2 real roots $\sigma_\pm$,

$$\sigma(k,+,1,R,P) > 0 \quad \Longleftrightarrow \quad R \;>\; R_0(k) \;=\; \frac{(k^2+\pi^2)^3}{k^2} \;.$$

Neutral curve:



↑ **Bifurcation !**

Minimum ↔ **critical wavenumber** $k_c \;=\; \pi/\sqrt{2} \;\simeq\; 2.22$

critical wavelength $\lambda_c \;=\; 2\pi/k_c \;=\; 2\sqrt{2} \;\simeq\; 2.83$

critical Rayleigh number $R_c \;=\; 27\pi^4/4 \;\simeq\; 657.5$

Thus an increase of 0.2% of R from 657 to 658 produces **'dramatic' effects**: the **system becomes unstable** ! A **bifurcation** is a **'catastrophe'** !..

2D RBT with slip BC: linear stability analysis: structures

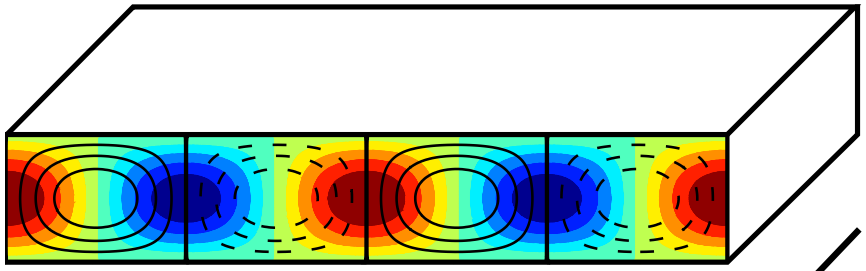
Complex critical mode: $V_{1c} \;=\; (-3i\pi/\sqrt{2},\, 1)\;\exp(ik_c x)\;\cos(\pi z)$

where we used the **normalization condition**

$$\theta(x=0, z=0) \text{ in } V_{1c} \;=\; 1$$

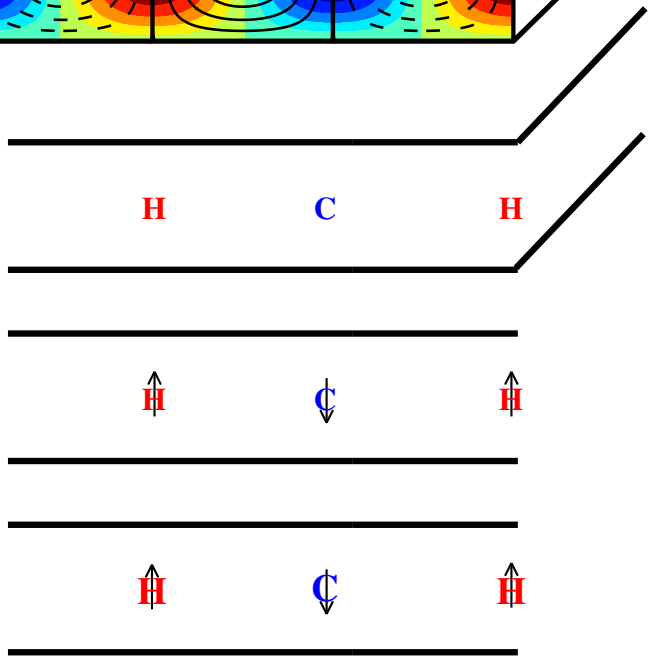
$\Rightarrow$ **real critical mode:** $V_{1r} \;=\; AV_{1c} + c.c. \;=\; A(3\sqrt{2}\pi\;\sin(k_c x),\, 2\cos(k_c x))\;\cos(\pi z)$

Streamlines and isotherms of θ :



These **convection rolls** show the **instability loop**:

- start with a modulation of temperature θ
- because $P^{-1}\partial_t v_z \;=\; R\theta$,
  this produces a modulation of vertical velocity v_z
- by advection, $\partial_t \theta \;=\; v_z$,
  this reinforces the initial modulation of temperature θ



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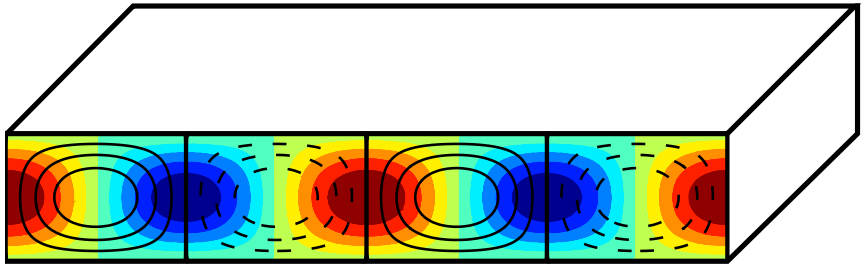
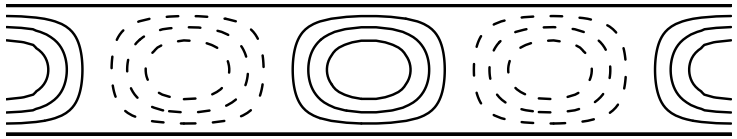
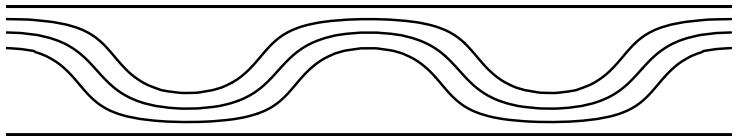


Isotherms of $T \;=\; T_{\text{cond}} + \theta$:

θ



T_{cond}

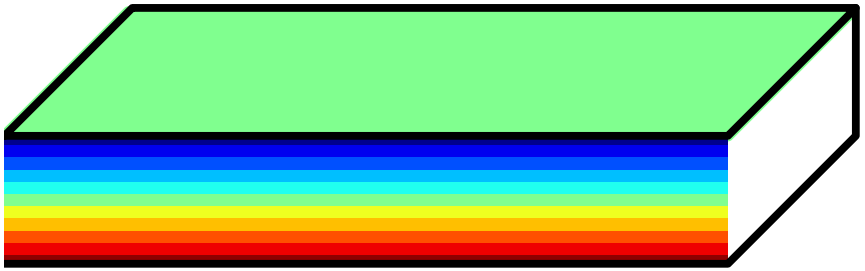


T

2D RBT with slip BC: linear stability analysis: structures

When one goes from the **static, conduction case**, for $R < R_c$,

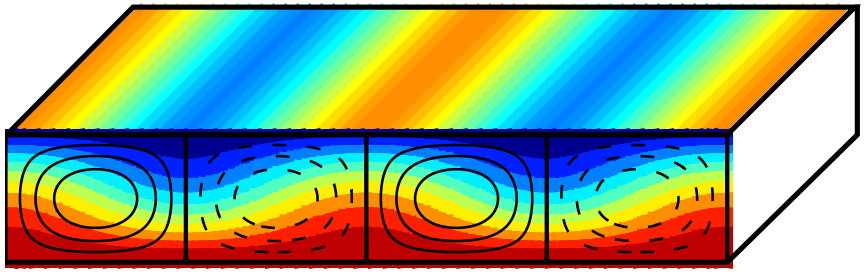
$$V = (\psi, \theta) = (0, 0)$$



to the **convection case**, for $R > R_c$,

$$V = (\psi, \theta) = V_{1r} = A\,(3\sqrt{2}\pi\ \sin(k_c x),\ 2\cos(k_c x))\ \cos(\pi z)$$

the temperature field averaged over z changes drastically:



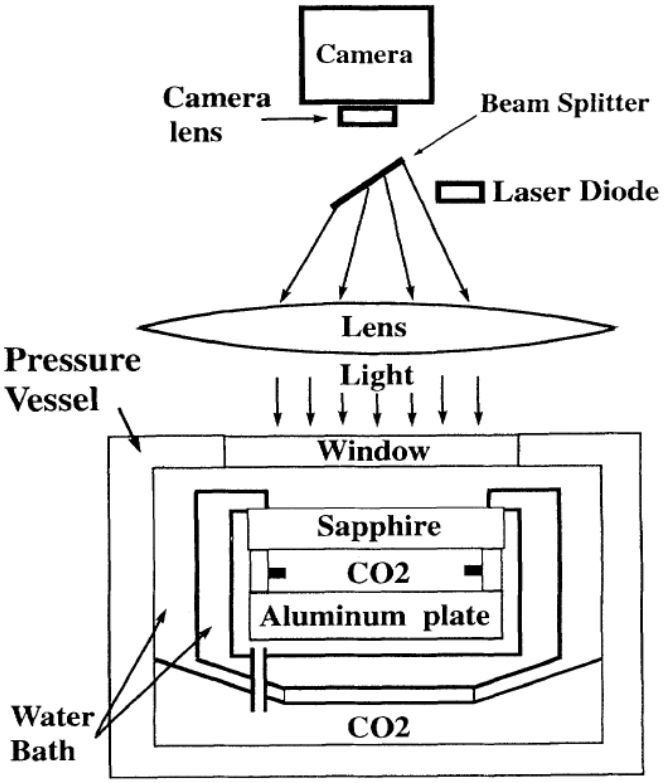
$\longleftrightarrow$ **'patterning instability'** that induces **'pattern formation' !**

(**'patterning bifurcation'**)

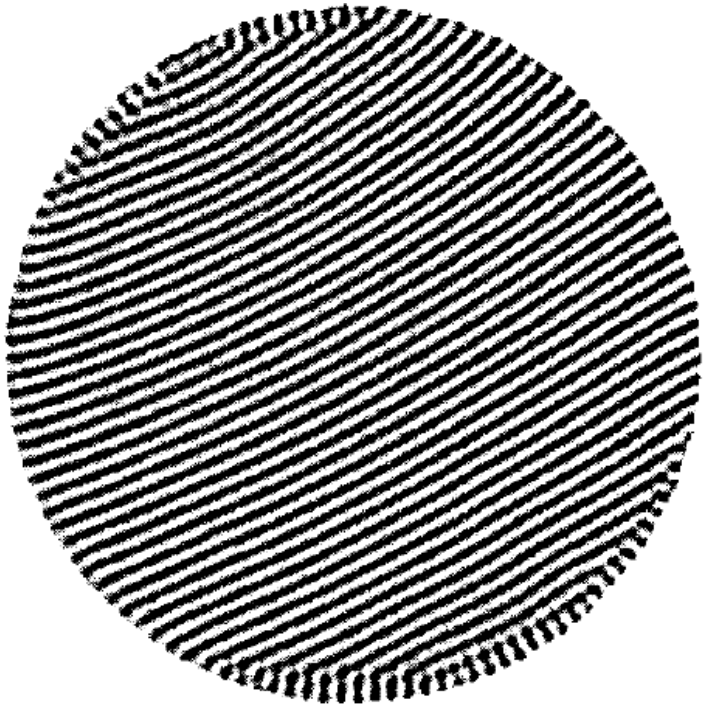
RBT: experimental result showing pattern formation !

RBT in CO_2 under pressure ($P = 0.93$), visualization with the **shadowgraph method** (the refractive index of the fluid depends on T...):

Apparatus:



Top view for $R = 1.04 R_c$:



[Hu, Ecke & Ahlers 1993 Convection near threshold for Prandtl numbers near 1. *Phys. Rev. E*]

2D RBT with slip BC: linear analysis: ex. 1.3 time dependence ?

Convection case, for $R > R_c$: $V = A(t)\,(3\sqrt{2}\pi\ \sin(k_c x),\, 2\cos(k_c x))\ \cos(\pi z)$,

$$A(t) = A_0\ \exp(\sigma t) \quad \text{with} \quad \sigma = \sigma_+ = \sigma(k_c, +, 1, R, P) = ?$$

σ_+ positive root of the characteristic equation ; with the other root σ_-,
and $D_1 = -\Delta = k_c^2 + \pi^2$,

$$\sigma_+ + \sigma_- = -(1+P)D_1 \ ,$$
$$\sigma_+\,\sigma_- = P(D_1^3 - R k_c^2)/D_1 \ .$$

Assume R close to R_c, $R = R_c(1+\epsilon)$ with the **bifurcation parameter** $\epsilon = R/R_c - 1 \ll 1$

$$\sigma_- \simeq -\sigma_1 \ , \quad \sigma_+ \simeq \epsilon/\tau_0 \longleftrightarrow \tau = 1/\sigma_+ \simeq \tau_0/\epsilon \longleftrightarrow$$ **critical slowing down !**

Characteristic time of the instability $\tau_0 = (2/(3\pi^2))\,(1+P^{-1})$.

In physical units, the field with the longest characteristic time controls the dynamics:

$$\tau_0^{\text{dimensional}} = \frac{2d^2}{3\pi^2\kappa}\,(1+P^{-1}) \simeq \tau_{\text{therm}} \quad \text{if} \quad P \gg 1 \quad \text{i.e.} \quad \tau_{\text{therm}} = \frac{d^2}{\kappa} \gg \tau_{\text{visc}} = \frac{d^2}{\nu}$$

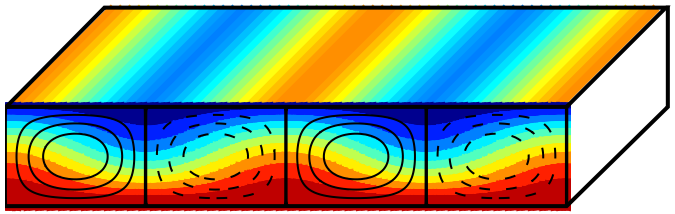
$$\tau_0^{\text{dimensional}} \simeq \frac{2d^2}{3\pi^2\kappa}\,P^{-1} \simeq \tau_{\text{visc}} \quad \text{if} \quad 1 \gg P \quad \text{i.e.} \quad \tau_{\text{visc}} \gg \tau_{\text{therm}}$$

'*Long-living systems slave short-living systems*' [Haken]

2D RBT with slip BC: linear analysis: ex. 1.3 time dependence

Convection case, for R close to R_c, $R = R_c(1+\epsilon)$ with the bifurcat° parameter $0 < \epsilon \ll 1$:

$$V \;=\; (\psi,\, \theta) \;=\; V_{1r} \;=\; A(t)\; (3\sqrt{2}\pi\;\; \sin(k_c x),\, 2\cos(k_c x))\;\; \cos(\pi z) \;\; :$$



$$A(t) \;=\; A_0\;\exp(\sigma t) \quad \textbf{grows exponentially !}$$

$$\sigma \;=\; \sigma_+ \simeq \frac{\epsilon}{\tau_0} \;\longleftrightarrow\; \tau \;=\; \frac{1}{\sigma_+} \;\simeq\; \frac{\tau_0}{\epsilon} \;\; \text{large (\textbf{critical slowing down !})},$$

with the characteristic time of the instability $\tau_0 \;=\; \dfrac{2}{3\pi^2}\,(1+P^{-1})$.

To obtain solutions for long times, we develop a **weakly nonlinear analysis**

- **in the vicinity of the bifurcation threshold**, the bifurcation parameter $0 < \epsilon \ll 1$;
- based on the use of the **basis of the linear modes**
  separating the '*long-living masters*' from the '*short-living slaves*' !

2D RBT with slip BC: basis of the linear modes

Work in a box with periodic BC under $x \mapsto x + \lambda_c$

$\Rightarrow$ the wavenumber $k \in \mathbb{K}$ with $\mathbb{K} = k_c\mathbb{Z}$.

Development in exponential Fourier series of x, trigonometric Fourier series of z of a general field

$$V = \sum_{k\in\mathbb{K}} \sum_{n\in\mathbb{N}^*} \widehat{V}(k,n) \ \exp(ikx) \ \sin(n\pi z + n\pi/2)$$

$\forall$ k and n, $(\Psi(k,+,n), \Theta(k,+,n))$ and $(\Psi(k,-,n), \Theta(k,-,n))$ form a basis of $\mathbb{C}^2$

$$\Rightarrow \quad \widehat{V}(k,n) = A(k,+,n) \ V_1(k,+,n) + A(k,-,n) \ V_1(k,-,n)$$

$\Rightarrow$ a general field

$$V = \sum_{k\in\mathbb{K}} \sum_{s=\pm} \sum_{n\in\mathbb{N}^*} A(k,s,n) \ V_1(k,s,n) = \sum_{\mathbf{q}} A(\mathbf{q}) \ V_1(\mathbf{q})$$

with $\mathbf{q} = (k,s,n) \in \mathbb{K} \times \{+,-\} \times \mathbb{N}^*$.

Basis of the linear modes: how to calculate the amplitudes ?

$$V = \sum_{k\in\mathbb{K}} \sum_{s=\pm} \sum_{n\in\mathbb{N}^*} A(k,s,n)\ V_1(k,s,n) = \sum_{\mathbf{q}} A(\mathbf{q})\ V_1(\mathbf{q})\ , \quad A(\mathbf{q}) = ?$$

▷ Introduce the **Hermitian inner product**

$$\langle V,\ U\rangle = \int_{x=0}^{\lambda_c}\int_{z=-1/2}^{1/2} V(x,z)\cdot U^*(x,z)\ \frac{dx}{\lambda_c}\ dz$$

▷ Define the **adjoint operators** $D^\dagger$ and $L^\dagger$ such that

$$\forall V,U,\quad \langle D\cdot V,\ U\rangle = \left\langle V,\ D^\dagger\cdot U\right\rangle \quad \text{and} \quad \langle L\cdot V,\ U\rangle = \left\langle V,\ L^\dagger\cdot U\right\rangle$$

V and U satisfying the BC of the problem

▷ The **adjoint eigenproblem**

$$\sigma^* D^\dagger\cdot U = L^\dagger\cdot U$$

has eigenvalues σ^* that are the complex conjugates of the ones of the direct eigenproblem

▷ To each direct (eigen)mode $V_1(\mathbf{q})$ of eigenvalue $\sigma(\mathbf{q})$
there corresponds an adjoint (eigen)mode $U_1(\mathbf{q})$ of eigenvalue $\sigma^*(\mathbf{q})$
with the same wavenumber k

Basis of the linear modes: how to calculate the amplitudes ?

$$V \;=\; \sum_{k\in\mathbb{K}} \sum_{s=\pm} \sum_{n\in\mathbb{N}^*} A(k,s,n)\; V_1(k,s,n) \;=\; \sum_{\mathbf{q}} A(\mathbf{q})\; V_1(\mathbf{q})\;, \quad A(\mathbf{q}) \;=\; ?$$

▷ $$\langle V,\, U\rangle \;=\; \int_{x=0}^{\lambda_c}\int_{z=-1/2}^{1/2} V(x,z)\cdot U^*(x,z)\;\frac{dx}{\lambda_c}\;dz$$

▷ $$\forall V,U,\quad \langle D\cdot V,\, U\rangle \;=\; \Big\langle V,\, D^\dagger\cdot U\Big\rangle \quad \text{and} \quad \langle L\cdot V,\, U\rangle \;=\; \Big\langle V,\, L^\dagger\cdot U\Big\rangle$$

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there corresponds an adjoint mode $U_1(\mathbf{q})$ of eigenvalue $\sigma^*(\mathbf{q})$ with the same k

▷ If k in $\mathbf{q}$ $\neq k'$ in $\mathbf{q}'$ then $\langle D\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; \langle L\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; 0$

▷ For $\mathbf{q}$ with the same wavenumber k, one has usually non degenerate eigenvalues:

$$\text{if } k \text{ in } \mathbf{q} \;=\; k \text{ in } \mathbf{q}' \;\text{ but }\; \mathbf{q}\neq\mathbf{q}'\;,\quad \sigma=\sigma(\mathbf{q}) \;\neq\; \sigma'=\sigma(\mathbf{q}')$$

▷ Consequently $\mathbf{q}\neq\mathbf{q}' \quad\Longrightarrow\quad \langle D\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; \langle L\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; 0$

Basis of the linear modes: how to calculate the amplitudes ?

$$V \;=\; \sum_{k\in\mathbb{K}} \sum_{s=\pm} \sum_{n\in\mathbb{N}^*} A(k,s,n)\; V_1(k,s,n) \;=\; \sum_{\mathbf{q}} A(\mathbf{q})\; V_1(\mathbf{q})\;, \quad A(\mathbf{q}) \;=\; ?$$

▷ $$\langle V,\, U\rangle \;=\; \int_{x=0}^{\lambda_c}\int_{z=-1/2}^{1/2} V(x,z)\cdot U^*(x,z)\;\frac{dx}{\lambda_c}\;dz$$

▷ $$\forall V,U,\quad \langle D\cdot V,\, U\rangle \;=\; \Big\langle V,\, D^\dagger\cdot U\Big\rangle \quad \text{and} \quad \langle L\cdot V,\, U\rangle \;=\; \Big\langle V,\, L^\dagger\cdot U\Big\rangle$$

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there corresponds an adjoint mode $U_1(\mathbf{q})$ of eigenvalue $\sigma^*(\mathbf{q})$ with the same k

▷ $$\mathbf{q}\neq\mathbf{q}' \quad\Longrightarrow\quad \langle D\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; \langle L\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q}')\rangle \;=\; 0$$

▷ The adjoint modes can be normalized such that

$$\forall\mathbf{q}\;,\quad \langle D\cdot V_1(\mathbf{q}),\, U_1(\mathbf{q})\rangle \;=\; 1 \quad\Longrightarrow\quad \boxed{A(\mathbf{q}) \;=\; \langle D\cdot V,\, U_1(\mathbf{q})\rangle}\;.$$

Physical interpretation of the adjoint modes: study of a (linearized) forcing problem

$$\boxed{D \cdot \partial_t V \;=\; L \cdot V \;+\; F}$$

for which we seek solutions

$$V \;=\; \sum_{\mathbf{q}} A(\mathbf{q},t)\; V_1(\mathbf{q})$$

$$\Rightarrow \quad \sum_{\mathbf{q}} \frac{dA}{dt}(\mathbf{q},t)\; D \cdot V_1(\mathbf{q}) \;=\; \sum_{\mathbf{q}} \sigma(\mathbf{q}) A(\mathbf{q},t)\; D \cdot V_1(\mathbf{q}) \;+\; F$$

Find the **'amplitude equations'** by 'projecting' onto $U_1(\mathbf{q})$

$$\Rightarrow \quad \frac{dA}{dt}(\mathbf{q},t) \;=\; \sigma(\mathbf{q}) A(\mathbf{q},t) \;+\; \langle F,\, U_1(\mathbf{q})\rangle$$

with the **forcing term**

$$\langle F,\, U_1(\mathbf{q})\rangle \;=\; \int_{x=0}^{\lambda_c} \int_{z=-1/2}^{1/2} F(x,z) \cdot U_1^*(\mathbf{q}; x,z)\; \frac{dx}{\lambda_c}\; dz$$

$\hookrightarrow$ the components of $U_1(\mathbf{q})$ measure the **'receptivity'** of the mode $V_1(\mathbf{q})$ to **perturbat°**, they are **'receptivity functions'**.

Application: ex. 1.4 : adjoint problem and adjoint modes in RBT

For Fourier modes in x, of wavenumber $k = mk_c$ with $m \in \mathbb{Z}^*$.

Local state vectors: $V = (\psi,\theta)$ and $U = (\psi_a,\theta_a)$.

BC: $\theta = \partial_x\psi = \partial_z^2\psi = \theta_a = \partial_x\psi_a = \partial_z^2\psi_a = 0$ if $z = \pm 1/2$

i.e. $\theta = \psi = \partial_z^2\psi = \theta_a = \psi_a = \partial_z^2\psi_a = 0$ if $z = \pm 1/2$.

Direct op.: $[D \cdot V]_\psi = P^{-1}(-\Delta\psi)$, $[L_R \cdot V]_\psi = -Rik\theta + \Delta(-\Delta\psi)$,

$$[D \cdot V]_\theta = \theta \ , \quad [L_R \cdot V]_\theta = \Delta\theta + ik\psi \ .$$

Inner product: $$\langle V, U\rangle = \int_{x=0}^{\lambda_c}\int_{z=-1/2}^{1/2} V(x,z)\cdot U^*(x,z)\,\frac{dx}{\lambda_c}\,dz.$$

$$\forall V,U, \quad \langle D\cdot V, U\rangle = \left\langle V, D^\dagger \cdot U\right\rangle \quad \text{and} \quad \langle L\cdot V, U\rangle = \left\langle V, L^\dagger \cdot U\right\rangle$$

$\hookrightarrow$ $$D^\dagger = D \ ,$$

$$[L^\dagger \cdot U]_\psi = -\Delta\Delta\psi_a - ik\theta_a, \quad [L^\dagger \cdot U]_\theta = \Delta\theta_a + Rik\psi_a \ .$$

Application: adjoint problem and adjoint modes in RBT

For Fourier modes in x, of wavenumber $k = mk_c$ with $m \in \mathbb{Z}^*$.

Local state vectors: $V = (\psi, \theta)$ and $U = (\psi_a, \theta_a)$.

Direct op.: $[D \cdot V]_\psi = P^{-1}(-\Delta\psi)$, $[L_R \cdot V]_\psi = -Rik\theta + \Delta(-\Delta\psi)$,

$$[D \cdot V]_\theta = \theta \ , \quad [L_R \cdot V]_\theta = \Delta\theta + ik\psi \ .$$

Adjoint op.: $[D \cdot U]_\psi = P^{-1}(-\Delta\psi_a)$, $[L_R^\dagger \cdot U]_\psi = -\Delta\Delta\psi_a - ik\theta_a$,

$$[D \cdot U]_\theta = \theta_a \ , \quad [L_R^\dagger \cdot U]_\theta = \Delta\theta_a + Rik\psi_a \ .$$

Adjoint problem: $\sigma \ D \cdot U = L_R^\dagger \cdot U$.

Ex. 1.5 : For $k = k_c = \pi/\sqrt{2}$, $R = R_c = 27\pi^4/4$, to the critical mode

$$V_{1c} = (-3i\pi/\sqrt{2}, 1) \ \exp(ik_c x) \ \cos(\pi z)$$

check that there corresponds a neutral adjoint critical mode U_{1c}
and calculate it with the normalization condition $\langle D \cdot V_{1c}, U_{1c} \rangle = 1$.

Solution:

$$U_{1c} = \frac{2}{1 + P^{-1}}(-i2\sqrt{2}/(9\pi^3), 1) \ \exp(ik_c x) \ \cos(\pi z) \ .$$

Weakly nonlinear analysis of 2D RBT with slip BC

We seek, for $\epsilon = R/R_c - 1 \ll 1$, an approximate solution of the nonlinear problem

$$\boxed{D \cdot \partial_t V = L_R \cdot V + N_2(V,V)} \qquad (*)$$

of the form

$$V = \sum_{\mathbf{q}} A(\mathbf{q},t)\ V_1(\mathbf{q}) \ .$$

Following Haken *'Long-living systems slave short-living systems'* we distinguish

- **active modes $\mathbf{q} = \mathbf{q}_c = (k_c, +,1)$ or $\mathbf{q}_c^* = (-k_c, +,1)$** which are long-living

$$\sigma(\mathbf{q},R) \sim \epsilon/\tau_0$$

- **passive modes $\mathbf{q} \neq \mathbf{q}_c, \mathbf{q}_c^*$** which are short-living (rapidly damped)

$$\sigma(\mathbf{q},R) < \sigma_1 < 0$$

and assume that (possibly after a short transient) **the active modes dictate the dynamics**:

$$\forall \mathbf{q} \ , \quad \frac{dA}{dt}(\mathbf{q},t) = O(\epsilon\ A(\mathbf{q},t)) \ ,$$

$$V = V_a + V_\perp \quad \text{with} \quad V_a = A_{1c} V_{1c} + c.c. \text{ the } \textbf{active modes}, V_a \ll 1 \ ,$$

$$V_\perp = \sum_{\mathbf{q} \neq \mathbf{q}_c, \mathbf{q}_c^*} A(\mathbf{q},t)\ V_1(\mathbf{q}) \text{ the } \textbf{passive modes}, V_\perp \ll V_a \ .$$

Weakly nonlinear analysis of 2D RBT with slip BC

We seek, for $\epsilon = R/R_c - 1 \ll 1$, an approximate solution of the nonlinear problem

$$\boxed{D \cdot \partial_t V = L_R \cdot V + N_2(V,V)} \qquad (*)$$

of the form $\qquad V = V_a + V_\perp$

with $V_a = A_{1c} V_{1c} + c.c.$ the **active modes**, of eigenvalue $\sigma(\mathbf{q},R) \sim \epsilon/\tau_0$,

$$V_\perp = \sum_{\mathbf{q}\neq\mathbf{q}_c,\mathbf{q}_c^*} A(\mathbf{q},t)\, V_1(\mathbf{q}) \text{ the } \textbf{passive modes}, \text{ of eigenvalue } \sigma(\mathbf{q},R) < \sigma_1 < 0 ,$$

$$\forall \mathbf{q} , \quad \frac{dA}{dt}(\mathbf{q},t) = O(\epsilon\, A(\mathbf{q},t)) .$$

'Amplitude equations' by projection of $(*)$ onto $U_1(\mathbf{q})$:

$$\frac{dA}{dt}(\mathbf{q},t) = \sigma(\mathbf{q},R)A(\mathbf{q},t) + \sum_{\mathbf{q}_1}\sum_{\mathbf{q}_2} A(\mathbf{q}_1,t)A(\mathbf{q}_2,t)\,\langle N_2(V_1(\mathbf{q}_1),V_1(\mathbf{q}_2)),\, U_1(\mathbf{q})\rangle$$

$\Rightarrow$ the **passive modes** are $O(A_{1c}^2)$ and can be calculated by **quasistatic elimination**

$$0 = \sigma(\mathbf{q},R)A(\mathbf{q},t) + \sum_{\mathbf{q}_1=\mathbf{q}_c,\mathbf{q}_c^*}\ \sum_{\mathbf{q}_2=\mathbf{q}_c,\mathbf{q}_c^*} A(\mathbf{q}_1,t)A(\mathbf{q}_2,t)\,\langle N_2(V_1(\mathbf{q}_1),V_1(\mathbf{q}_2)),\, U_1(\mathbf{q})\rangle$$

$$\Longleftrightarrow \quad 0 = L_R \cdot V_\perp + N_2(V_a,V_a) . \qquad (**)$$

WNLA of 2D slip RBT: quasistatic elimination of the passive modes

$$V_{1c} \;=\; (-3i\pi/\sqrt{2},\,1)\;\exp(ik_c x)\;\cos(\pi z)\;,$$
$$A \;=\; A_{1c}\;,$$
$$V_a \;=\; A\,(\psi_1,\,\theta_1) \;=\; AV_{1c} \;+\; c.c. \;=\; A(3\sqrt{2}\pi\;\sin(k_c x),\,2\cos(k_c x))\;\cos(\pi z)\;,$$

$$[L_R\cdot V]_\psi \;=\; -R\partial_x\theta + \Delta(-\Delta\psi)\;,\;\; [N_2(V,V)]_\psi \;=\; P^{-1}\big[\partial_x\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_z\big)-\partial_z\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_x\big)\big]\;, \quad \text{(Vort)}$$

$$[L_R\cdot V]_\theta \;=\; \Delta\theta \;+\; \partial_x\psi\;,\;\; [N_2(V,V)]_\theta \;=\; -\,\mathbf{v}\cdot\boldsymbol{\nabla}\theta\;, \quad \text{(HE)}$$

$$0 \;=\; L_R\cdot V_\perp \;+\; N_2(V_a,V_a)\;. \quad (**)$$

Ex. 1.6 : Show with Mathematica that

$$[N_2(V_a,V_a)]_\psi \;=\; 0\;,\;\; [N_2(V_a,V_a)]_\theta \;=\; B\;\sin(2\pi z)$$

with B a real number, $B \;=\; 3\pi^3 A^2$.

Then, solve $(**)$ and explain the physics behind, with drawings.

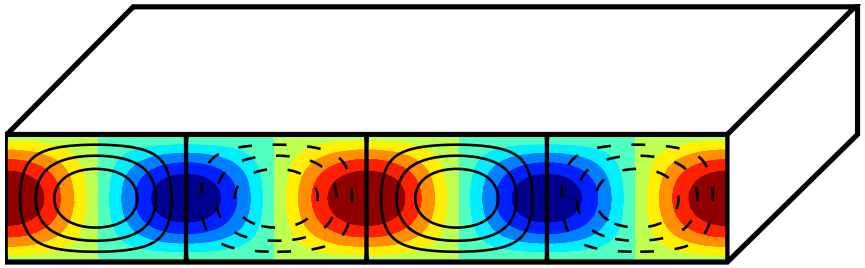
▷ Useful Mathematica commands: `ExpToTrig`, `D`, `Simplify`, `DSolve`, `Replace`.

Solution:

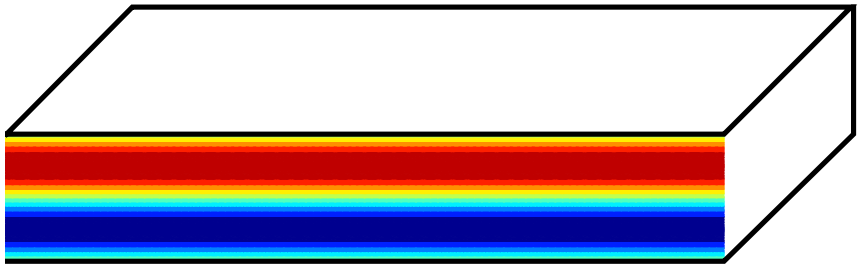
$$V_\perp \;=\; A^2\,(0,\,\Theta_2) \quad \text{with} \quad \Theta_2 \;=\; \frac{3\pi}{4}\;\sin(2\pi z)\;.$$

WNLA of 2D slip RBT: physics of the passive mode

Streamlines (iso-ψ_1) and isotherms of θ_1 :



Isotherms of $\Theta_2 = \frac{3\pi}{4} \sin(2\pi z)$:



This **mode** due to the advection of θ_1 by $\mathbf{v}_1$ describes the **enhanced heat transfer** as measured by the (global) **Nusselt number** (**ex. 1.7**)

$$Nu = \frac{\Phi_{\text{heat with conduction \& convection}}}{\Phi_{\text{heat with conduction only}}} = 1 - \langle \partial_z \theta \rangle_x = 1 - A^2 (\partial_z \Theta_2)_{z=\pm 1/2} = 1 + \frac{3\pi^2}{2} A^2 .$$

Pb : we do not know the value of the amplitude A !

WNLA of 2D slip RBT: amplitude equation

We seek, for $\epsilon = R/R_c - 1 \ll 1$, an approximate solution of the nonlinear problem

$$\boxed{D \cdot \partial_t V = L_R \cdot V + N_2(V,V)} \qquad (*)$$

of the form $$V = V_a + V_\perp + h.o.t.$$

with $V_a = AV_{1c} + c.c.$ the **active modes**, of eigenvalue $\sigma(\mathbf{q},R) \sim \epsilon/\tau_0$,

$V_\perp = A^2 V_{20}$ with $V_{20} = (0, \Theta_2)$ the **passive mode**, of eigenvalue $\sigma(\mathbf{q},R) < \sigma_1 < 0$.

'Amplitude equation' for A by projection of $(*)$ onto :

$$U_1(\mathbf{q}_c) = U_{1c} = \frac{2}{1+P^{-1}}(-i2\sqrt{2}/(9\pi^3), 1) \ \exp(ik_c x) \ \cos(\pi z)$$

$$\Rightarrow \quad \frac{dA}{dt} = \sigma(\mathbf{q}_c,R)A + \langle N_2(V,V), U_1(\mathbf{q}_c)\rangle$$

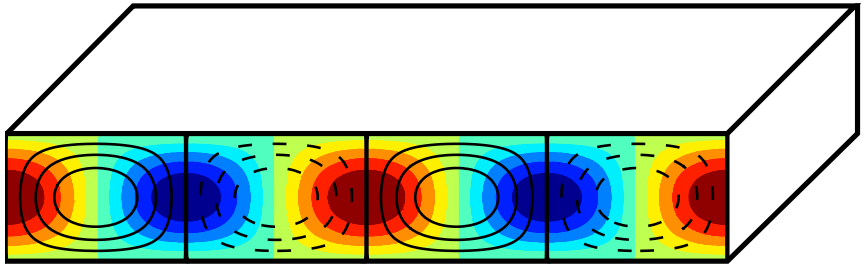
Nonlinear terms in $N_2(V,V)$ that have a nonzero projection on $U_1(\mathbf{q}_c)$ are **'resonant'**.

Ex. 1.8 : Compute the **resonant terms** in $N_2(V,V)$ - explain their **physics** !..

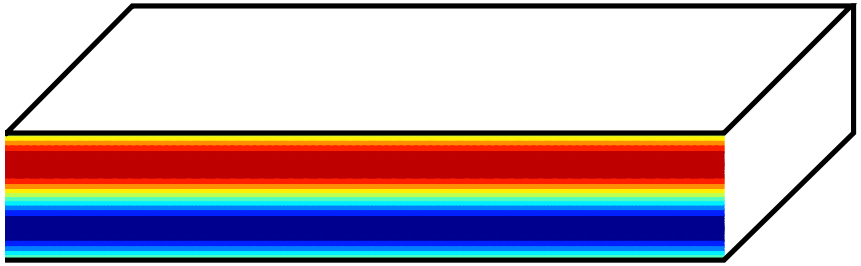
Solution: Resonant terms in $[N_2(V_a,V_\perp)]_\theta \propto -A^3 \cos(k_c x) \cos(\pi z) \cos(2\pi z)$.

WNLA of 2D slip RBT: resonant modes at order A^3

Streamlines (iso-ψ_1) and isotherms of θ_1 :

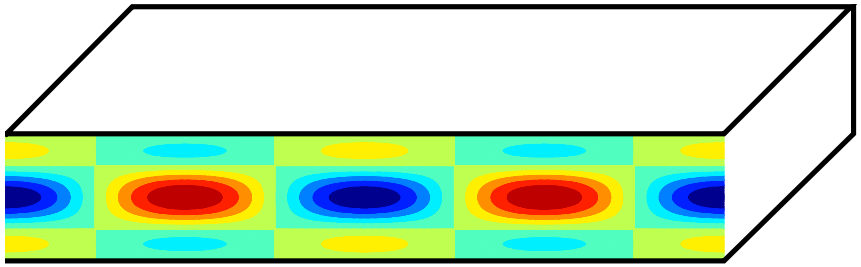


Isotherms of $\Theta_2 = \frac{3\pi}{4} \sin(2\pi z)$:



Isotherms of $N_{\theta 3}$:

$$N_{\theta 3} \propto -\cos(k_c x)\cos(\pi z)\cos(2\pi z)$$

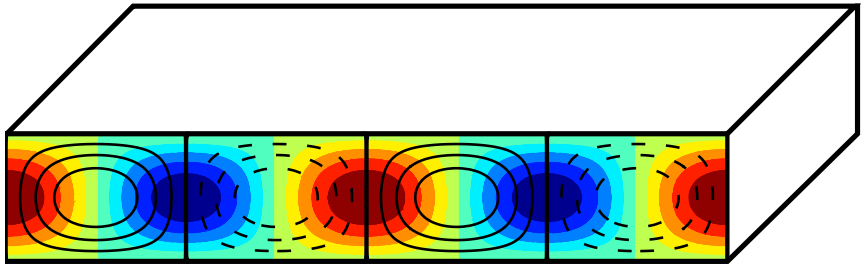


This mode due to the advection of Θ_2 by $\mathbf{v}_1$ feedbacks on the critical mode...

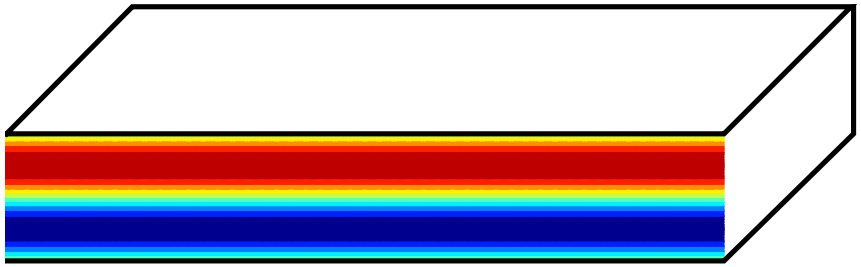
Ex. 1.9 : analyze this feedback !..

WNLA of 2D slip RBT: resonant modes at order A^3

Streamlines (iso-ψ_1) and isotherms of θ_1 :

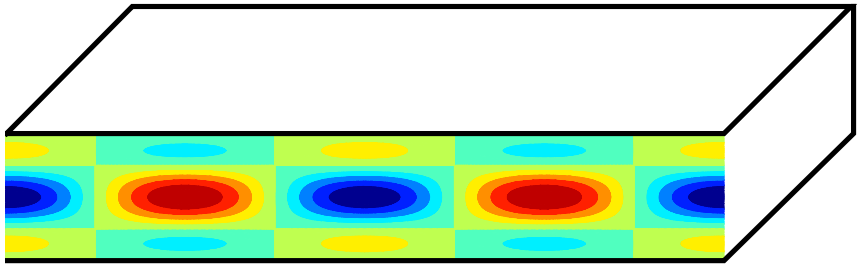


Isotherms of $\Theta_2 = \frac{3\pi}{4} \sin(2\pi z)$:



Isotherms of $N_{\theta 3}$:

$$N_{\theta 3} \propto -\cos(k_c x)\cos(\pi z)\cos(2\pi z)$$



This mode due to the advection of Θ_2 by $\mathbf{v}_1$ traduces a **saturation effect**, as measured by the **saturation coefficient** in the amplitude equation for A :

$$\frac{dA}{dt} = \frac{\epsilon}{\tau_0} A - gA^3 \quad \text{with} \quad g = -\langle N_2(V_{1c}, V_{20}),\, U_{1c}\rangle$$

Ex. 1.9 : Compute g. ▷ Useful Mathematica commands: `Conjugate`, `Integrate`.

WNLA of 2D slip RBT: supercritical pitchfork bifurcation

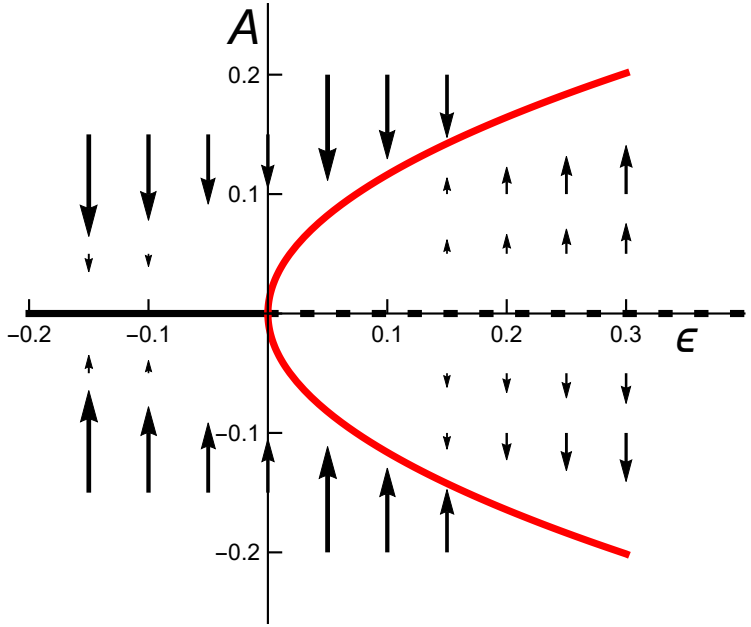
The **amplitude equation**

$$\frac{dA}{dt} = \frac{\epsilon}{\tau_0} A - gA^3 \quad \text{with} \quad g > 0$$

has always a **trivial solution** $A = 0$ corresponding to the **conduction state**;
this solution is **stable for** $\epsilon < 0$, **unstable for** $\epsilon > 0$.

For $\epsilon > 0$ there appear 2 **symmetric stable solutions** (stable 'fixed points') which correspond to **convection**

$$A = \pm\sqrt{\epsilon/(\tau_0 g)}\,.$$



WNLA of 2D slip RBT: supercritical pitchfork bifurcation

For $\epsilon > 0$, the **amplitude equation**

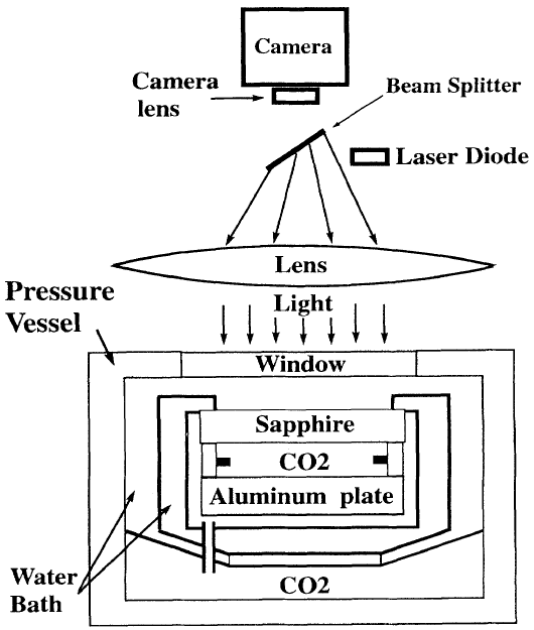
$$\frac{dA}{dt} = \frac{\epsilon}{\tau_0} A - gA^3 \quad \text{with} \quad g > 0$$

has 2 **symmetric stable solutions** which correspond to **convection** $A = \pm\sqrt{\epsilon/(\tau_0 g)}$.

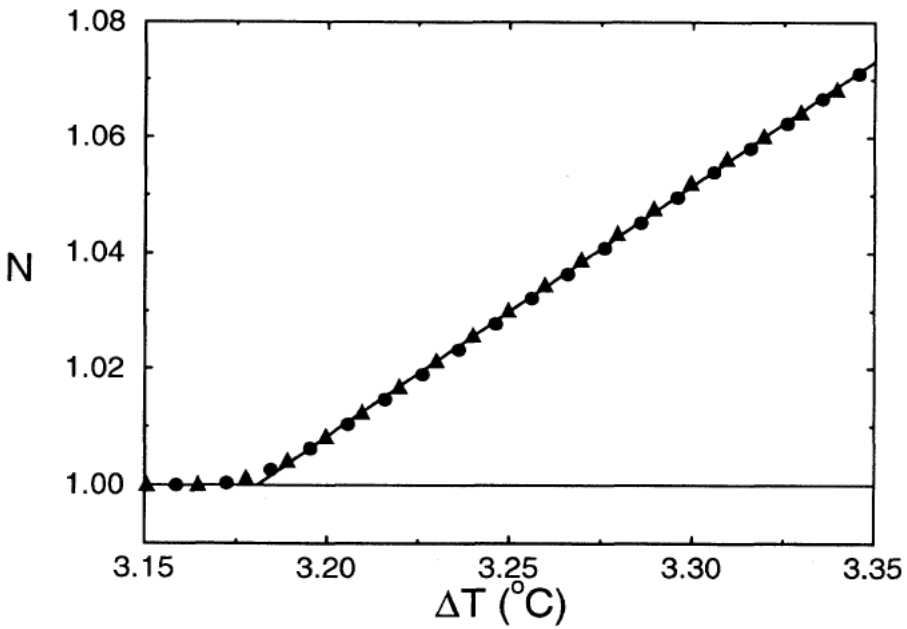
$$\Rightarrow \quad \text{Nusselt number } Nu = 1 + \frac{3\pi^2}{2} A^2 = 1 + a\epsilon \quad \text{with} \quad a > 0$$

as confirmed, at least semi-quantitatively, by the experiments of Hu et al. 1993:

Apparatus:



Nusselt number vs δT :



2D RBT with more realistic no-slip BC: short review - linear A.

- With these BC

$$\partial_x \psi = \partial_z \psi = \theta = 0 \quad \text{if} \quad z = \pm 1/2$$

the **linear stability analysis** cannot be done analytically.

- An efficient **numerical method** is the **spectral one**: the eigenfunctions of the Fourier modes are searched as a sum of simple polynomial functions,

$$\Psi(z) = \sum_{n=1}^{N} \Psi_n F_n(z) \quad \text{with} \quad F_n(z) = (1/2 - z)^2 \, (z + 1/2)^2 \, T_{2n-2}(2z) \, ,$$

$$\Theta(z) = \sum_{n=1}^{N} \Theta_n f_n(z) \quad \text{with} \quad f_n(z) = (1/2 - z) \, (z + 1/2) \, T_{2n-2}(2z) \, ,$$

T_n the nth Chebyshev polynomial of the first kind.

- By evaluating the linear eq. at the Gauss-Lobatto collocation points

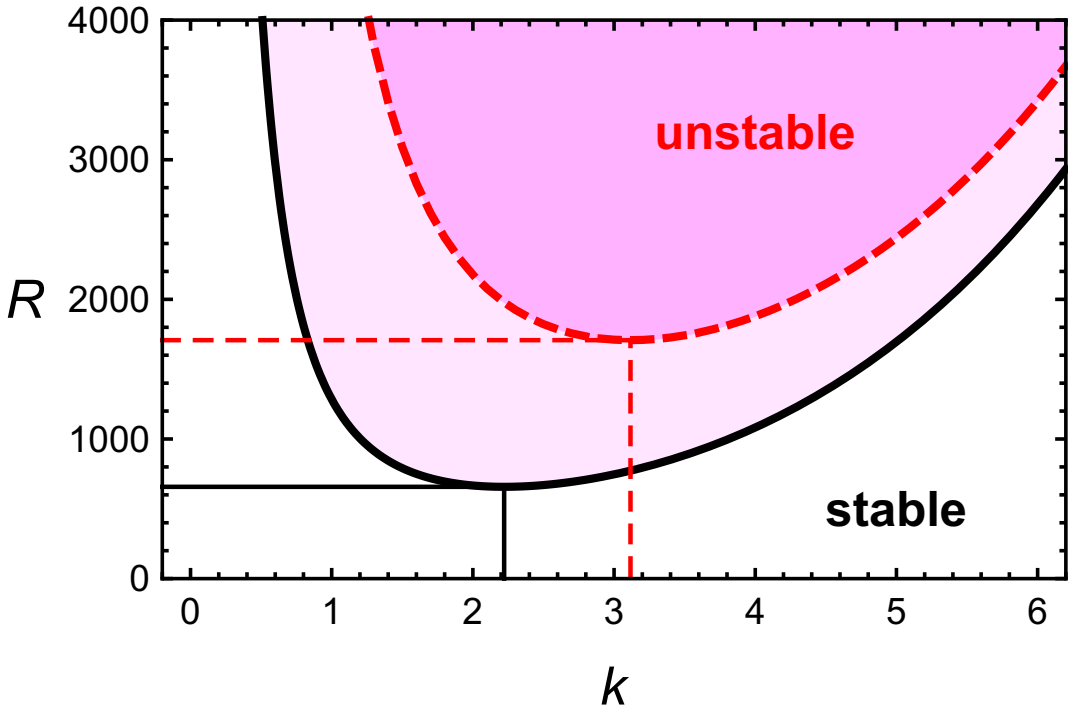
$$z_m = \cos[m\pi/(2N+1)]/2 \quad \text{for} \quad m \in \{1, 2, \cdots, N\} \, ,$$

one obtains a linear eigensystem for the vector $(\Psi_1, \cdots, \Psi_N, \Theta_1, \cdots, \Theta_N)$

- As soon as $N \gtrsim 3$, the most relevant eigenvalue is converged, see ex. 1.10...

2D RBT with more realistic no-slip BC: short review - linear A.

- Thus a **numerical linear analysis** can be performed.
  Results shown here with the dashed lines,
  to compare to the continuous lines for the **slip BC**:



Stabilizing effect of the no-slip BC : $R_{cNS} = 1708 > R_{cS} = 657.5$

Smaller, square rolls with no-slip BC : $\lambda_{cNS} = 2.01 < \lambda_{cS} = 2.83$

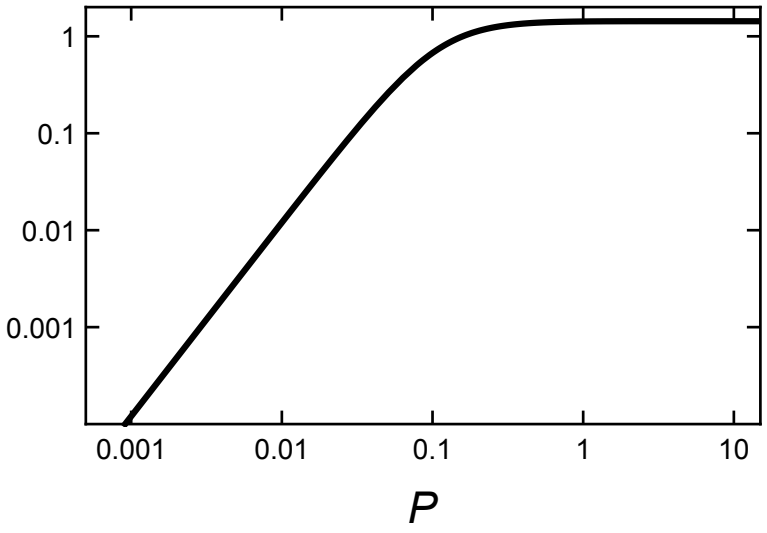
2D RBT with more realistic no-slip BC: short review - WNL A.

- Thus a **numerical linear and weakly nonlinear analysis** can be performed...
- The Nusselt number in the weakly nonlinear regime

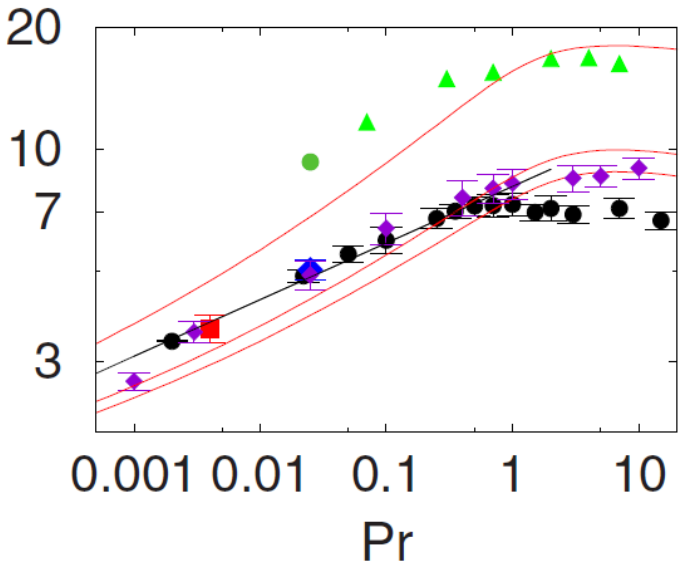
$$Nu = 1 + (0.699 + 0.00472P^{-1} + 0.00832P^{-2})^{-1}\epsilon + O(\epsilon^2)$$

depends strongly on the Prandtl number,
see $(Nu-1)/\epsilon$ vs P :

similar to the behaviour of **turbulent RBC**
here Nu at $R = 6\ 10^5$:



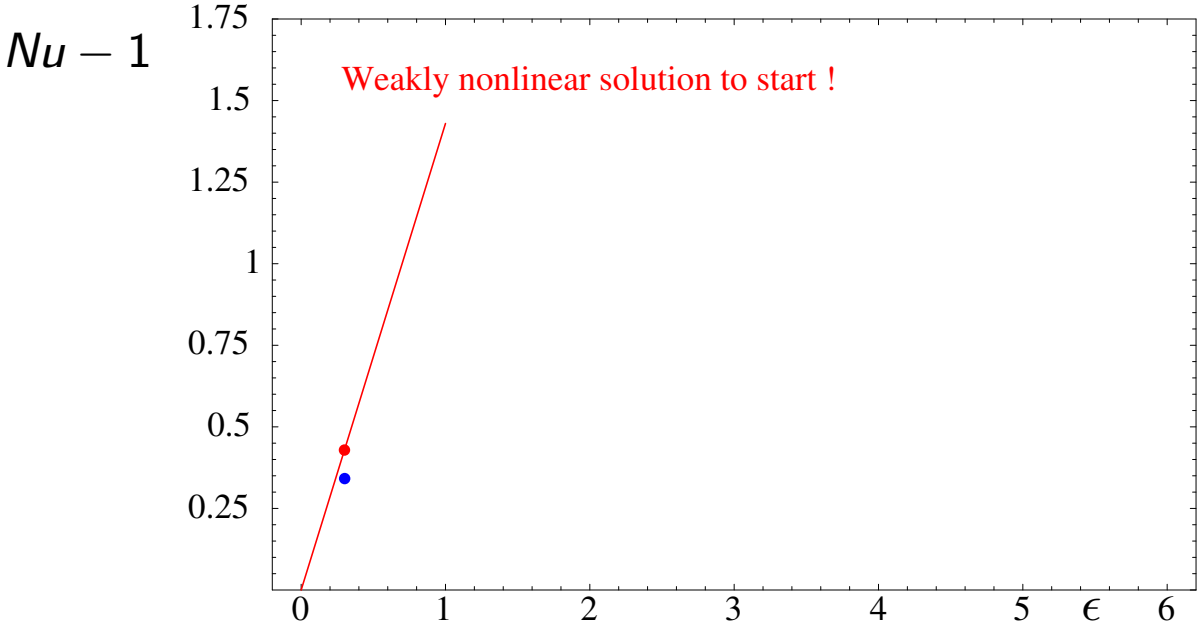
[Schlüter, Lortz & Busse 1965]

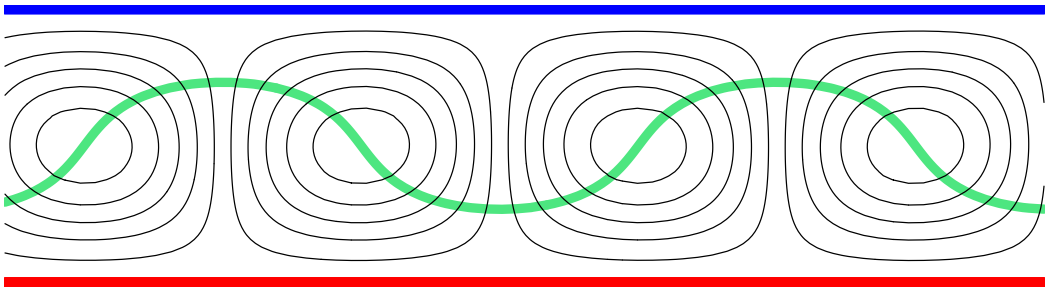


[Ahlers, Grossmann & Lohse 2009]

2D RBT with more realistic no-slip BC: short review

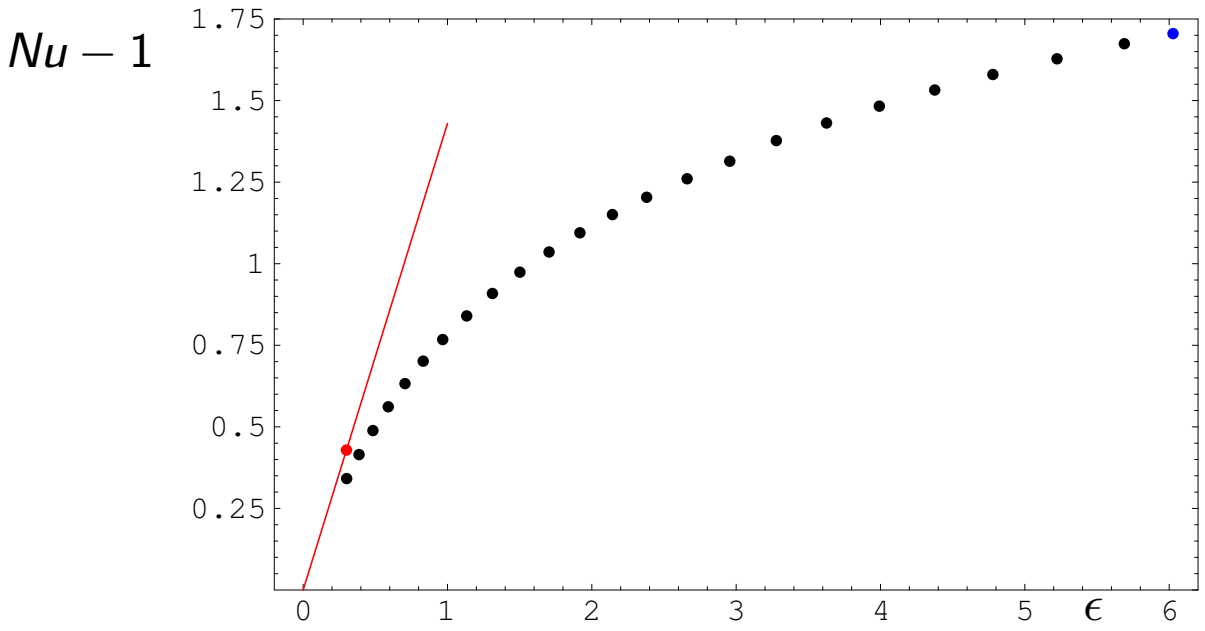
- Thus a **numerical linear and weakly nonlinear analysis** can be performed...
- The **spectral method** can also work in the **highly nonlinear regime**, when coupled to **continuation methods**...

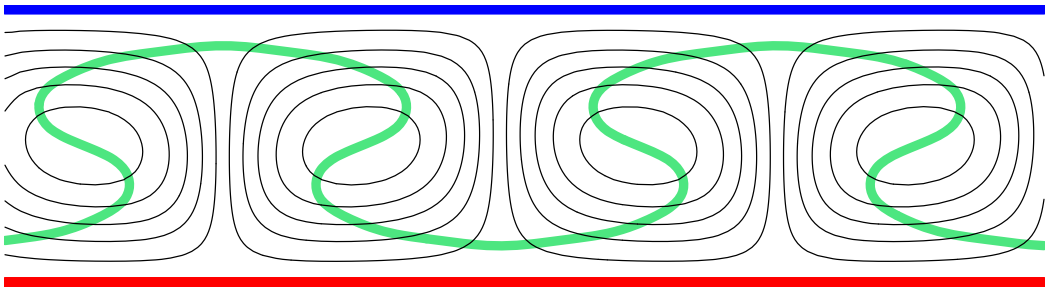




2D RBT with more realistic no-slip BC: short review

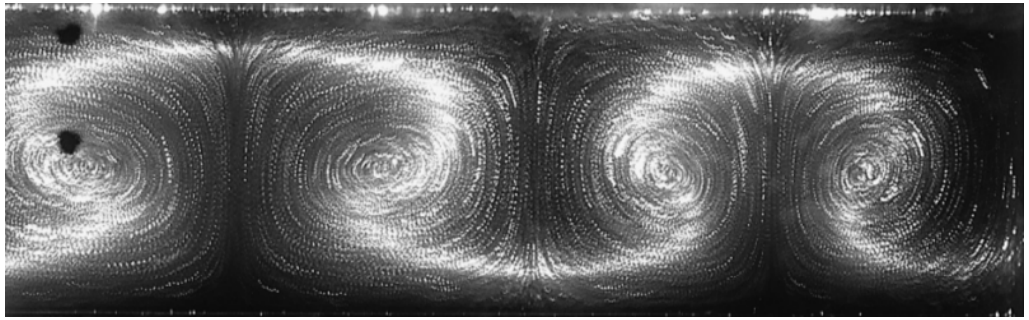
- Thus a **numerical** **linear and weakly nonlinear analysis** can be performed...
- The **spectral method** can also work in the **highly nonlinear regime**, when coupled to **continuation methods**...



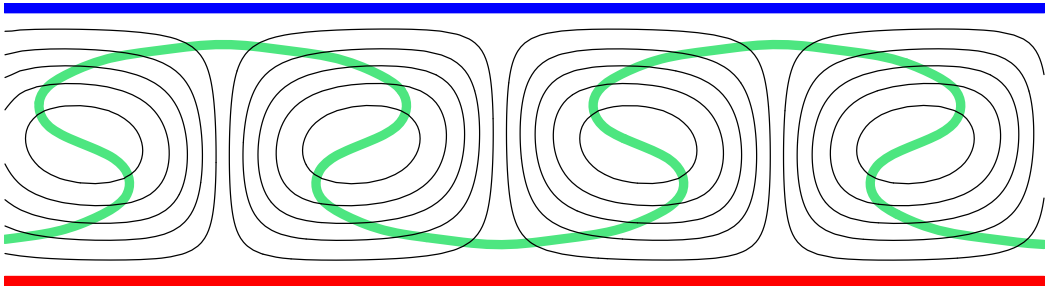


2D RBT with more realistic no-slip BC: short review

- The numerical solution obtained with the **spectral method** compares well with this experimental photo:



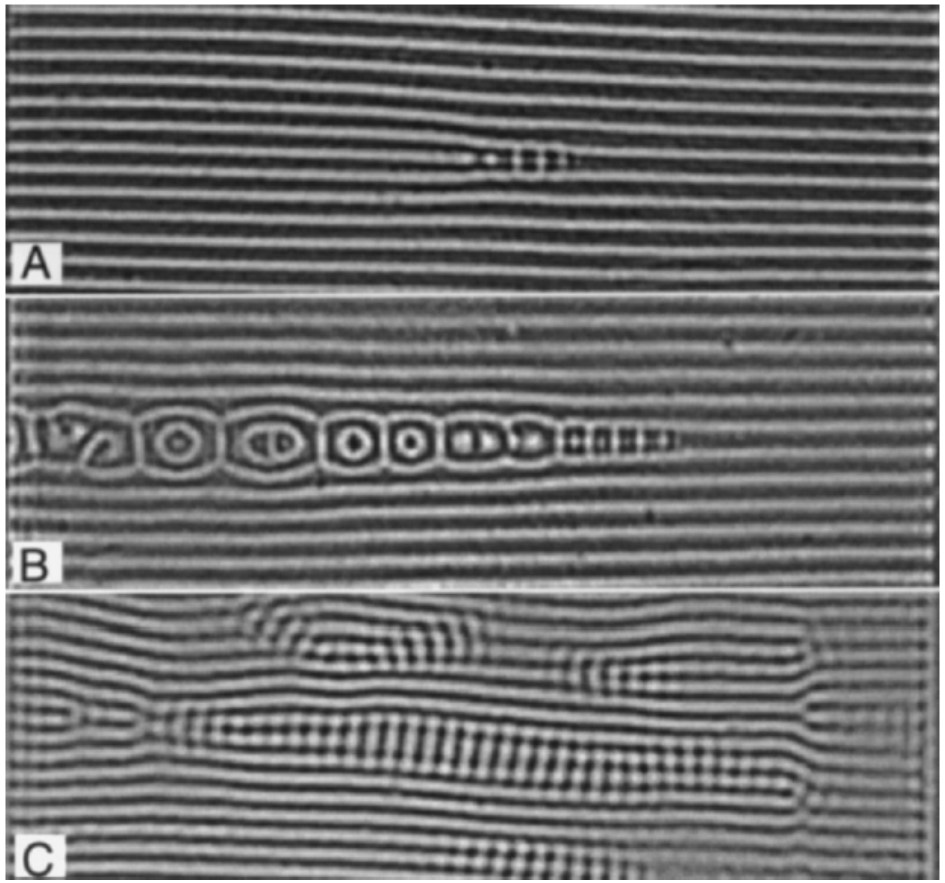
[Stasiek 1997 Thermochromic liquid crystals and true colour image processing in heat transfer and fluid-flow research. *Heat and Mass Transfer*]



This confirms for this case - fluid = glycerol, $P = 12.5\ 10^3$, $R = 12\ 10^3$ - the relevance of the Oberbeck - Boussinesq equations !

3D RBT with realistic no-slip BC

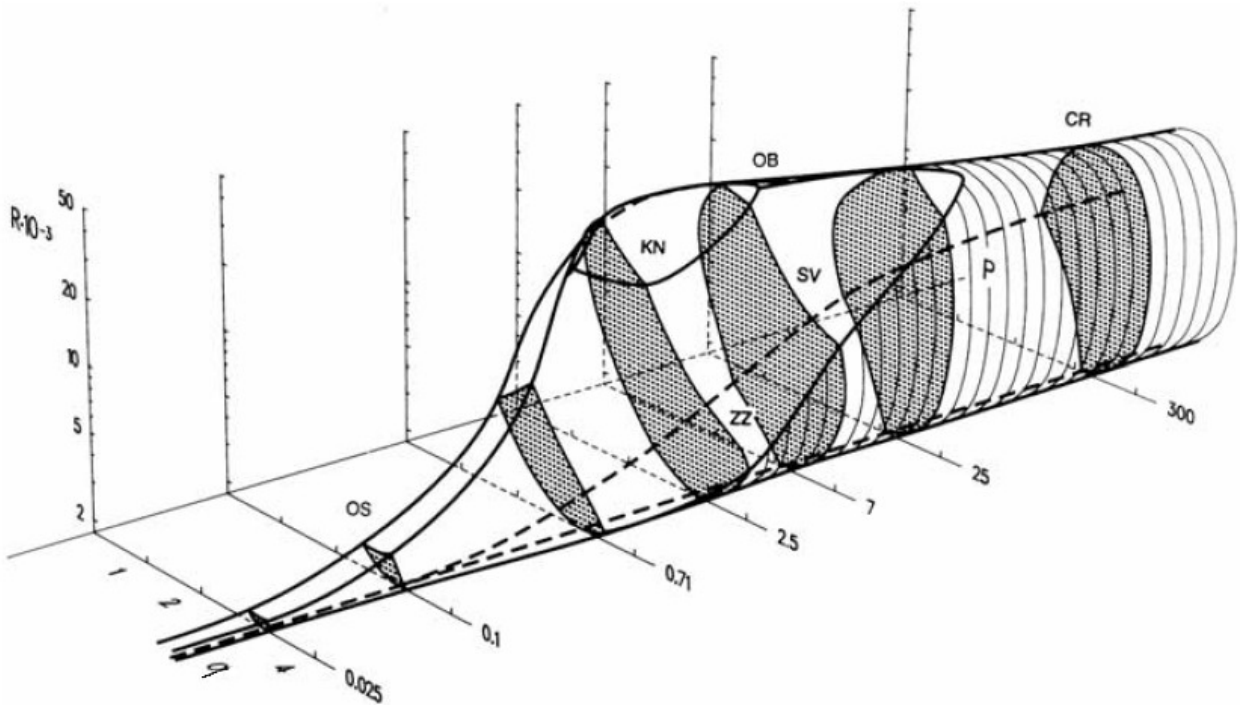
- When (k,R) departs from (k_c,R_c), **secondary instabilities** 'destroy' the rolls, for instance, the **cross-roll instability**:



[Plapp 1997 PhD Thesis. *Cornell University*]

3D RBT with realistic no-slip BC: globally supercritical scenario of transition

- These **secondary instabilities** that 'destroy' the rolls have been systematically studied by Busse and coworkers, to define the domain of stability of rolls in the (k,P,R) space:



[Busse 2003 The Sequence-of-Bifurcations Approach towards Understanding Turbulent Fluid Flow. *Surv. in Geophys.*]

- After **tertiary instabilities**, etc... this leads at high R to **turbulent flows**...

Coming back to slip RBT... Lorenz system - Chaos !

Considering the WNL solution

$$V \;=\; V_a \;+\; V_\perp \;+\; h.o.t.$$

with $V_a \;=\; AV_{1c} \;+\; c.c. \;=\; A\,\left(3\sqrt{2}\pi\;\sin(k_c x),\, 2\cos(k_c x)\right)\;\cos(\pi z)\;,$

$V_\perp \;=\; A^2 V_{20} \;=\; A^2\;\left(0,\, \frac{3\pi}{4}\sin(2\pi z)\right)\;,$

gave to Edward Lorenz, an American mathematician & meteorologist,
the idea to study thermoconvection flows of the form

$$\psi \;=\; A\;\sin(k_c x)\;\cos(\pi z)\;,\quad \theta \;=\; B\;\cos(k_c x)\;\cos(\pi z) + C\;\sin(2\pi z)\;.$$

By inserting this ansatz into the OB eq., neglecting 'h.o.t.', renormalizing time and the amplitudes (A, B, C) to define new ones (X, Y, Z), he obtained the **Lorenz system**

$$\left\{\begin{array}{rcll} P^{-1}\dot{X} & = & Y - X & \\ \dot{Y} & = & (\epsilon-1)X - Y & - XZ \\ \dot{Z} & = & -\frac{8}{3}Z & + XY \end{array}\right. .$$

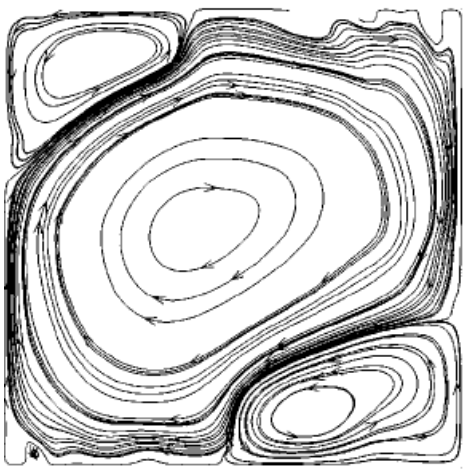
[Lorenz 1963 Deterministic nonperiodic flow. *J. Atm. Sci.*]

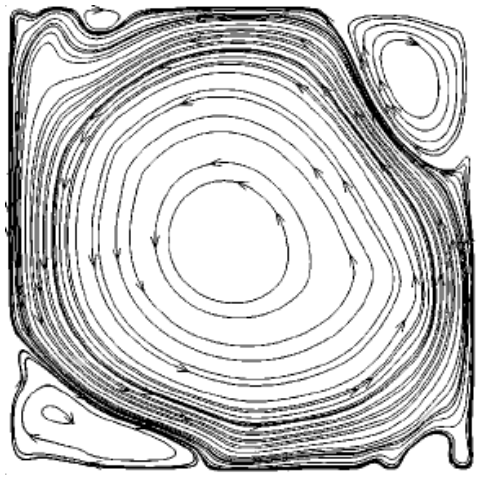
This system solved numerically shows **sensitive dependence on the initial condition**...
i.e. **chaos !** See e.g. https://www.youtube.com/watch?v=FYE4JKAXSfY...

What about no-slip RBT in a confined geometry - 2D square cell ?

At high $R = 5\ 10^7$ ($P = 4.3$), one expects one (non-perfect) roll only,

rotating clockwise **or** counter-clockwise...



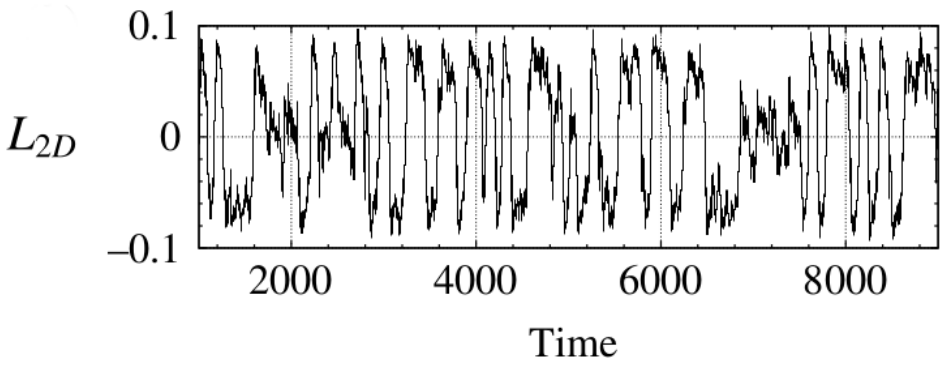


[Podvin & Sergent 2015 A large-scale investigation of wind reversal in a square Rayleigh-Bénard cell. *J. Fluid Mech.*]

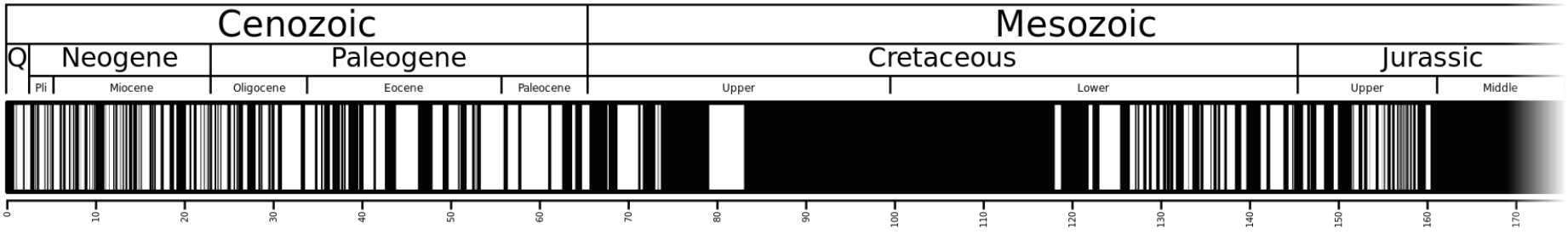
Thank you Bérengère P. for the movie !

What about no-slip RBT in a confined geometry - 2D square cell ?

The flow switches chaotically (**'flow reversals'**) between these two 'states', as traduced by this time series of the global angular momentum along y :



Phenomenologically, this behaviour is reminiscent of the **'geomagnetic field reversals'** that have been recorded in the 'frozen' ferromagnetic minerals of consolidated sedimentary deposits or cooled volcanic flows on land... This is also **chaos !..**
And, there are also **thermoconvection phenomena** 'behind' the geomagnetic field !..



[Geomagnetic reversal. *Wikipedia*]