Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

http://emmanuelplaut.perso.univ-lorraine.fr/t2t

Session	Date	Content
1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
→ 2 -	06/10	**RB** Thermoconvection: linear & weakly nonlinear stability analysis
3 -	13/10	**RB** Thermoconvection: nonlinear phenomena
4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB = Rayleigh-Bénard   **OSF** = Open Shear Flows

- I propose **2 possible homeworks (HW)** today, you will choose between **3 possible HW**...

Transition to spatial complexity in Rayleigh-Bénard thermoconvection

Today: session 2

1 The system - The equations and (stress-free) slip boundary conditions

2 Linear stability analysis - Structures - Patterning bifurcation

Next thursday: session 3

2 Linear stability analysis - Time dependence

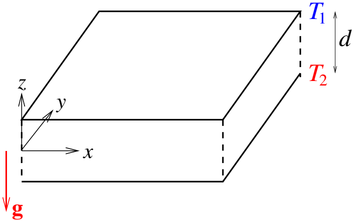
3 Weakly nonlinear analysis: towards the amplitude equation...

- ▷ The direct and adjoint mode bases... at order A...
- ▷ Quasistatic elimination of the passive mode at order A^2 - Nusselt number
- ▷ Resonant terms at order A^3...
- ▷ Amplitude equation, supercritical bifurcation

4 Results in the highly nonlinear regime, for other BC, and another geometry...

Rayleigh-Bénard Thermoconvection: dimensionless model

- Unit of length = thickness d
- Unit of time = heat diffus° time $\tau_{\text{therm}} = \dfrac{d^2}{\kappa}$
- Unit of velocity = $V = \dfrac{d}{\tau_{\text{therm}}} = \dfrac{\kappa}{d}$
- Unit of temperature = $\delta T = T_2 - T_1$



Introduce a dimensionless perturbat° of temperature θ, s.t. the dimensionless temperature

$$T' = T'_0 - z' + \theta$$

$\Rightarrow$ dimensionless **Oberbeck - Boussinesq equations**

$$\mathrm{div}\mathbf{v} = 0 \; , \qquad \text{(MC)}$$

$$P^{-1}\frac{d\mathbf{v}}{dt} = R\theta\,\mathbf{e}_z - \nabla p + \Delta\mathbf{v} \; , \qquad \text{(NS)}$$

$$\frac{d\theta}{dt} = \Delta\theta + v_z \; , \qquad \text{(HE)}$$

with the **Rayleigh number** $R =$ and the **Prandtl number** $P =$.

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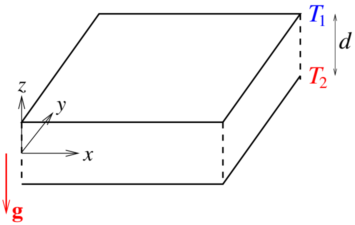

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with the **Rayleigh number** $R = \alpha\,\delta T\,g\,d^3/(\kappa\nu)$ and the **Prandtl number** $P = \nu/\kappa$.

Rayleigh-Bénard Thermoconvection: dimensionless model

OB equations:

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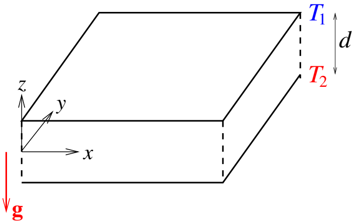

Isotropy of the problem in the horizontal plane $\Rightarrow$ focus on **2D** xz **solutions**

$$\mathbf{v} \;=\; v_x(x,z,t)\;\mathbf{e}_x \;+\; v_z(x,z,t)\;\mathbf{e}_z \;, \quad \theta \;=\; \theta(x,z,t)\;.$$

Solve (MC)

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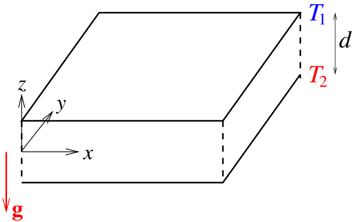

Isotropy of the problem in the horizontal plane $\Rightarrow$ focus on **2D** xz **solutions**

$$\mathbf{v} = v_x(x,z,t)\ \mathbf{e}_x + v_z(x,z,t)\ \mathbf{e}_z , \quad \theta = \theta(x,z,t) .$$

Solve (MC) by introducing a **streamfunction** ψ such that

$$\mathbf{v} = \mathbf{curl}(\psi\ \mathbf{e}_y) = (\nabla\psi)\times\mathbf{e}_y = -(\partial_z\psi)\ \mathbf{e}_x + (\partial_x\psi)\ \mathbf{e}_z .$$

Rayleigh-Bénard Thermoconvection: dimensionless model

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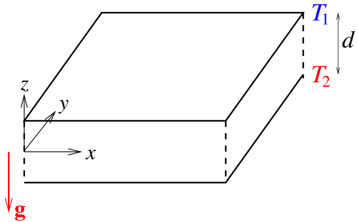

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Eliminate p in (NS)

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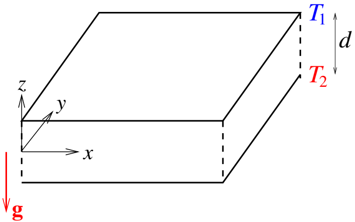

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Eliminate p in (NS) by considering $\mathbf{curl}(\text{NS})\cdot\mathbf{e}_y$ i.e. the **vorticity equation**:

$$P^{-1}\partial_t(-\Delta\psi) \;+\; P^{-1}\left[\partial_z\left(\mathbf{v}\cdot\nabla v_x\right) - \partial_x\left(\mathbf{v}\cdot\nabla v_z\right)\right] \;=\; -R\partial_x\theta \;+\; \Delta(-\Delta\psi) \;. \qquad \text{(Vort)}$$

RBT: 2D *xz* model

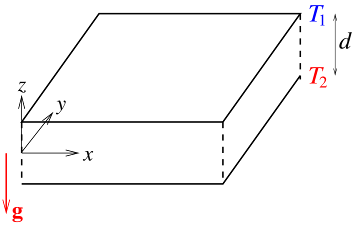
Local state vector: $V \;=\; (\psi,\theta)$ s. t.

$$\mathbf{v} \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z \;,$$

$$T \;=\; T_0 \;-\; z \;+\; \theta \;,$$

obeys the system of coupled PDE

$$\boxed{D\cdot\partial_t V \;=\; L_R\cdot V \;+\; N_2(V,V)}\;.$$



$$[D\cdot\partial_t V]_\psi \;=\; P^{-1}(-\Delta\partial_t\psi)\;,\quad [L_R\cdot V]_\psi \;=\; -R\partial_x\theta \;+\; \Delta(-\Delta\psi)\;, \qquad \text{(Vort)}$$

$$[N_2(V,V)]_\psi \;=\; P^{-1}\big[\partial_x\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_z\big) - \partial_z\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_x\big)\big]\;, \qquad \text{(Vort)}$$

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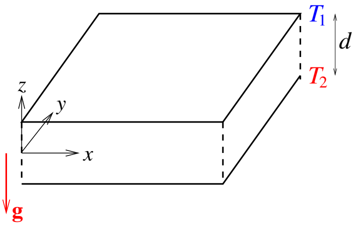

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Boundary conditions on θ :

RBT: 2D xz model

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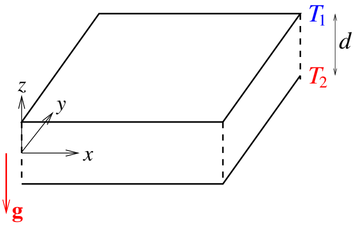

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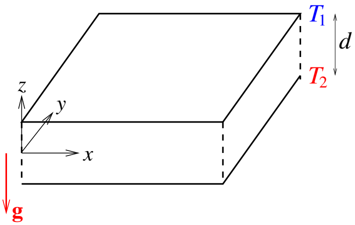

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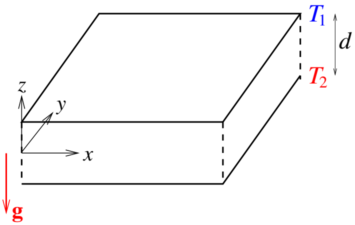

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Boundary conditions on ψ i.e. $\mathbf{v}$: **slip without stress ('stress-free'):**

$$v_z = 0 \quad \text{and} \quad \tau_{xz} = \partial_z v_x = 0 \iff \partial_x\psi = \partial_z^2\psi = 0 \quad \text{if} \quad z = \pm 1/2 \,.$$

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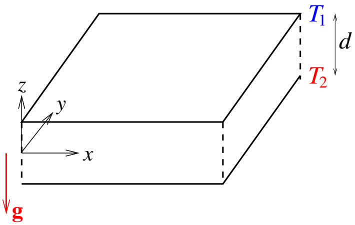

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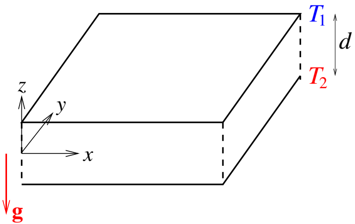
Extended geometry in the xy plane: (no BC or) periodic BC under $x \mapsto x + L$.

RBT 2D xz model with slip BC: linear stability analysis

Local state vector: $V = (\psi,\theta)$ s. t.



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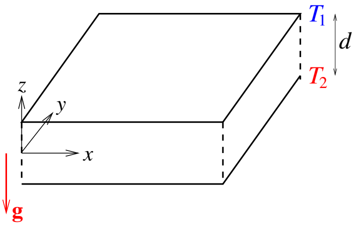

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Ex. 1.1: Normal mode analysis: the solution of the initial value problem is the superposition of normal modes that are Fourier modes in $\exp(ikx)$,

$$V = V_1(k,N)\ \exp[\sigma(k,N)\ t] \quad \text{with} \quad V_1(k,N) = (\widehat{\Psi}(z),\ \widehat{\Theta}(z))\ \exp(ikx)\ ,$$

k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes,

$\sigma(k,N)$ the **temporal eigenvalue**.

RBT 2D *xz* model with slip BC: linear stability analysis

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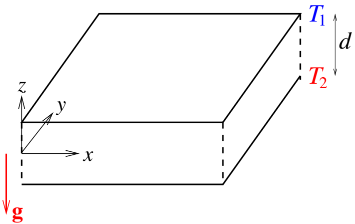

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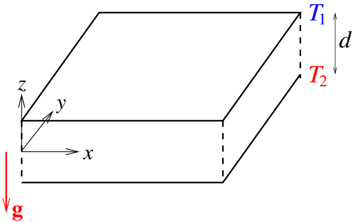
k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes, $\sigma(k,N)$ the **temporal eigenvalue**.

Boundary conditions: $\widehat{\Psi} = \widehat{\Psi}'' = 0 \;, \quad \widehat{\Theta} = 0 \quad \text{if} \quad z = \pm 1/2 \;.$

RBT 2D xz model with slip BC: normal mode analysis



Local state vector: $V = (\psi,\theta)$ s. t.

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$$\boxed{\sigma D\cdot V = L_R\cdot V} \ ,$$

$$[D\cdot V]_\psi = P^{-1}(-\Delta\psi) \ , \quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi) \ , \qquad \text{(VortE)}$$

$$[D\cdot V]_\theta = \theta \ , \quad [L_R\cdot V]_\theta = \Delta\theta + v_z \ , \qquad \text{(HE)}$$

$$\partial_x\psi = \partial_z^2\psi = \theta = 0 \quad \text{if} \quad z = \pm 1/2 \ .$$

Ex. 1.1: **<u>Generalized eigenvalue problem</u> solved by normal modes analysis: most relevant normal modes** are Fourier modes in $\exp(ikx)$ and have a z-profile in $\cos(\pi z)$,

$$V = V_1(k,N)\ \exp[\sigma(k,N)\ t] \quad \text{with} \quad V_1(k,N) = (\Psi,\ \Theta)\ \exp(ikx)\ \cos(\pi z) \ ,$$

k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes,
$\sigma(k,N)$ the **temporal eigenvalue**.

They satisfy the boundary conditions: $\widehat{\Psi} = \widehat{\Psi}'' = 0 \ , \quad \widehat{\Theta} = 0 \quad \text{if} \quad z = \pm 1/2 \ .$

RBT 2D *xz* model with slip BC: normal mode analysis

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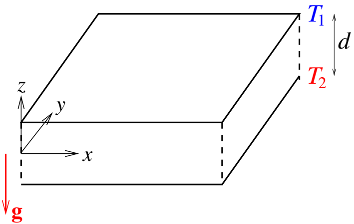

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RBT 2D *xz* model with slip BC: normal mode analysis

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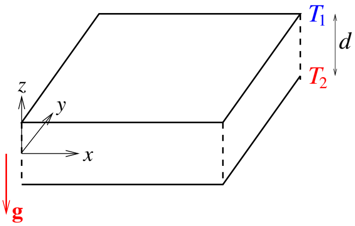

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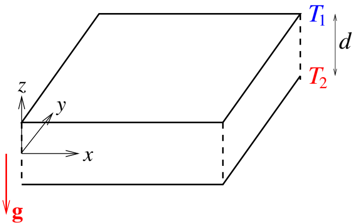

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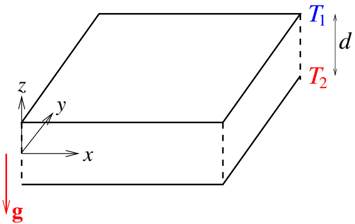

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$\sigma_+\sigma_- = P(D_1^3 - Rk^2)/D_1 > 0$ for small R $\leftrightarrow$ $\sigma_\pm < 0$

$\sigma_+\sigma_- = P(D_1^3 - Rk^2)/D_1 < 0$ for large R $\leftrightarrow$ $\sigma_- < 0,\ \sigma_+ > 0$

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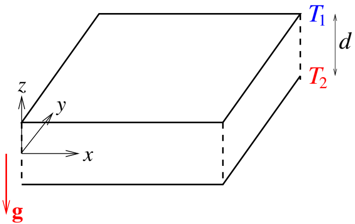

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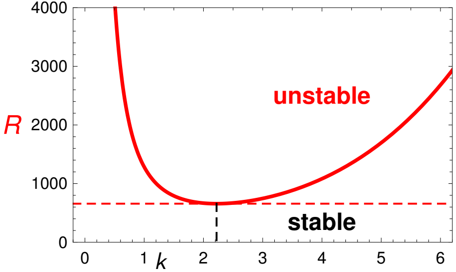
$\sigma_+\sigma_- = P(D_1^3 - Rk^2)/D_1 < 0$ for large R $\leftrightarrow$ $\sigma_- < 0,\ \sigma_+ > 0$ **instability !**

RBT 2D *xz* model with slip BC: normal mode analysis: results !

This characteristic equation for the **temporal eigenvalue** has 2 real roots $\sigma_\pm$,

$$\sigma(k,+,1,R,P) > 0 \quad \Longleftrightarrow \quad R \; > \; R_0(k) \; = \; (k^2+\pi^2)^3/k^2 \; .$$

Neutral curve :



↑ **Bifurcation !**

Minimum ↔ **critical wavenumber** $k_c \;=\; \pi/\sqrt{2} \;\simeq\; 2.22$

critical wavelength $\lambda_c \;=\; 2\pi/k_c \;=\; 2\sqrt{2} \;\simeq\; 2.83$

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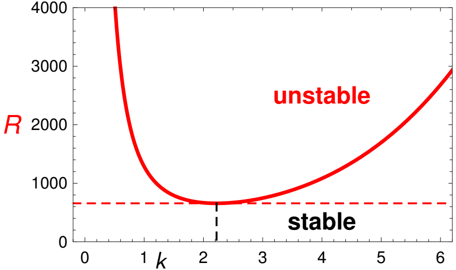

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Thus an increase of 0.2% of R from 657 to 658 produces **'dramatic' effects**: the **system becomes unstable** ! Some say that a **bifurcation** is a **'catastrophe'** !..

RBT 2D xz model with slip BC: normal mode analysis... However !

$$\boxed{\sigma\ D \cdot V \ = \ L_R \cdot V}$$

Ex. 1.1: Eigenproblem solved by normal modes analysis: most relevant normal modes

$$V \ = \ V_1(k,\pm)\ \exp[\sigma(k,\pm,1,R,P)\ t] \quad \text{with} \quad V_1(k,\pm) \ = \ (\Psi,\ \Theta)\ \exp(ikx)\ \cos(\pi z)\ ,$$

k = **horizontal wavenumber** $\neq 0$.

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- Quid of x-homogeneous modes with $k = 0$?
- Are there other modes with a more complex z-dependence ?

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$\hookrightarrow$ **ex. 1.2: general linear stability analysis** = **first of 3 possible homeworks !..**

- x-homogeneous modes are 'not dangerous' !
- More general x-dependent modes with $k \neq 0$:

$$V \ = \ V_1(k,\pm,n) \ = \ (\Psi,\ \Theta)\ \exp(ikx)\ \sin(n\pi z + n\pi/2)$$

$$n \leftrightarrow \text{dependence on } z\ ,\ \pm \leftrightarrow 2 \text{ modes at fixed } k \text{ and } n.$$

- Modes with $n > 1$ 'not dangerous' at 'low' values of R ?..

2D RBT with slip BC: linear stability analysis: structures

Complex critical mode: $V_{1c} =$

where we used the **normalization condition**

$$\theta(x=0, z=0) \text{ in } V_{1c} = 1$$

$\Rightarrow$ **real critical mode:** $V_{1r} =$

Streamlines and isotherms of θ **?**

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Complex critical mode: $V_{1c} = (-3i\pi/\sqrt{2},\, 1)\ \exp(ik_c x)\ \cos(\pi z)$

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$\Rightarrow$ **real critical mode:** $V_{1r} \;=\; AV_{1c} + c.c. \;=\; A(3\sqrt{2}\pi\;\sin(k_c x),\, 2\cos(k_c x))\;\cos(\pi z)$

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