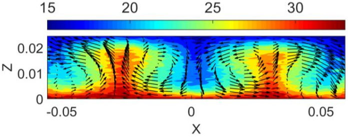
Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

Emmanuel Plaut

How does the **flow** in a **(closed or open) fluid system**
change from **laminar** to **complex or turbulent** as a control parameter is changed ?

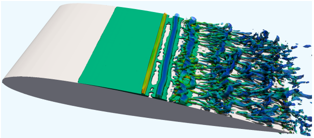
Fluid physics:

part 1: **thermoconvection**
closed heated fluid systems



[Leclerc & Métivier]

part 2: **aerodynamics**
open shear flows



[Tangermann & Klein]

Methods:

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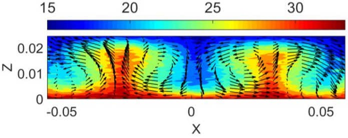

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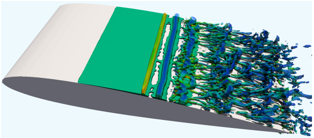

[Tangermann & Klein]

Methods: **linear** then **weakly nonlinear stability analysis**
= **bifurcation theory** or **'catastrophe theory'**

Analytical calculations in part 1 vs numerical computations in part 2 with a 'spectral method'... and Mathematica !

Transition to (spatio-temporal complexity and) turbulence in thermoconvection & aerodynamics

http://emmanuelplaut.perso.univ-lorraine.fr/t2t

Session	Date	Content
→ 1 -	29/09	Thermoconvection: phenomena, equations, differentially heated cavity, cavity heated from below = **RB** cavity, linear stability analysis
2 -	06/10	**RB** Thermoconvection: linear & weakly nonlinear stability analysis
3 -	13/10	**RB** Thermoconvection: nonlinear phenomena
4 -	20/10	Aerodynamics of **OSF**: linear stability analysis
5 -	27/10	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 -	10/11	Aerodynamics of **OSF**: nonlinear phenomena
	24/11	**Examination**

RB = Rayleigh-Bénard **OSF** = Open Shear Flows

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Follow up module in January & February 2023: **Turbulence & Wind Energy**

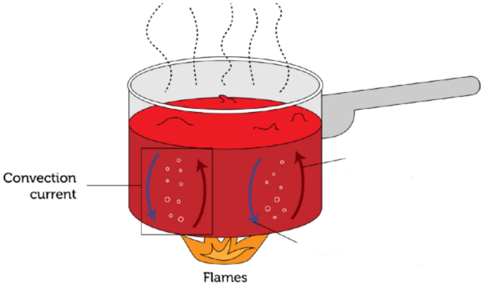
http://emmanuelplaut.perso.univ-lorraine.fr/twe

1st part of this module:
Transition to turbulence, or, to spatio-temporal complexity, in natural thermoconvection

- Fluids in non-isothermal situations

- temperature gradients buoyancy forces
  may drive **natural thermoconvection** = **heat-driven flows and transfers** !

- This happens in the **kitchen**...



1^{st} part of this module: Transition to turbulence, or, to spatio-temporal complexity, in natural thermoconvection

- Fluids in non-isothermal situations have a density ρ that depends on the temperature T, $\rho = \rho(T)$ which often $\downarrow$ as $T \uparrow$.
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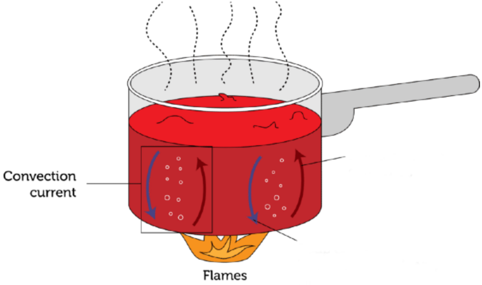

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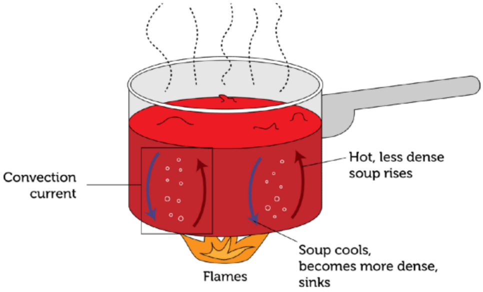

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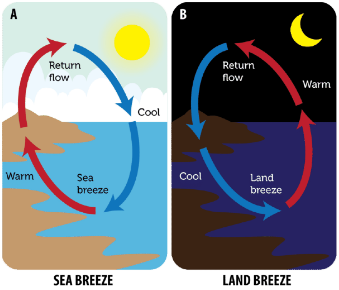

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The question is: **how thermoconvection comes in and develops** ?

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Today - session 1

- **Natural thermoconvection**:
  introduction, equations, example of the **differentially heated cavity**
- **Rayleigh-Bénard system** = **cavity heated from below**:
  linear stability analysis with slip boundary conditions

Natural thermoconvection: equations

- Fluids in non-isothermal situations have a density ρ that depends on T

$$\rho \;=\; \rho(T) \;.$$

- If temperature gradients exist, in a gravity field, **buoyancy forces** part of $\rho\mathbf{g}$ may drive **natural thermoconvection**.

- Equations of motion

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- Equations of motion of a **Newtonian fluid**:

$$\rho \frac{d\mathbf{v}}{dt} \; = \; \rho[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] \; = \; \rho\mathbf{g} \; - \; \boldsymbol{\nabla} p \; + \; \eta\Delta\mathbf{v} \; , \qquad \text{(NS)}$$

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- **Hydrostatic conduction solutions**: $\mathbf{v} = \mathbf{0} \Longrightarrow \boldsymbol{\nabla}T \parallel \mathbf{g}$.

  Hence $\boldsymbol{\nabla}T$ not vertical $\Longrightarrow$ **thermoconvection flows always develop**.

Natural thermoconvection: equations under the Oberbeck-Boussinesq approximations

- Fluids in non-isothermal situations have a density ρ that depends on T, under the 1st OB approximation, linearly:

$$\rho \;=\; \rho_0 \; [1 - \alpha(T - T_0)]$$

with ρ_0 the reference density, T_0 the reference temperature,
α the small thermal expansion coefficient.

- Equations of motion under the **OB approximations**:

$$\rho_0 \frac{d\mathbf{v}}{dt} \;=\; \rho_0[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] \;=\; \rho\mathbf{g} \;-\; \boldsymbol{\nabla} p \;+\; \eta\Delta\mathbf{v} \;, \qquad \text{(NS)}$$

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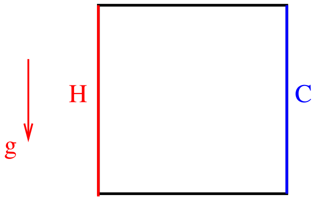
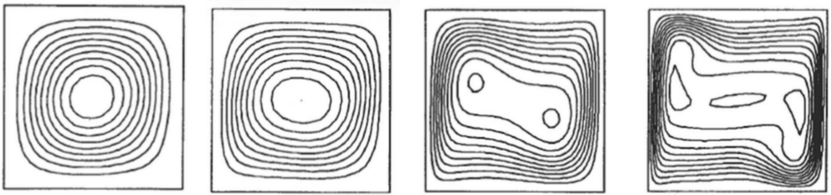
Thermoconvection in a differentially heated cavity



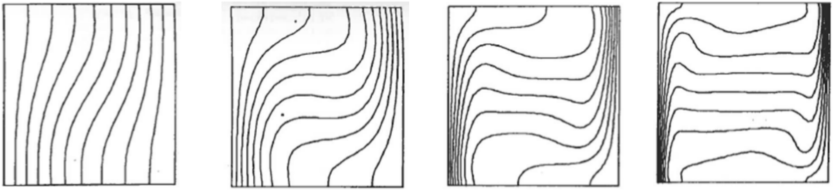
$$T = T_0 + \frac{\delta T}{2} \qquad\qquad T = T_0 - \frac{\delta T}{2}$$

Steady thermoconvection in a 2D differentially heated cavity

Streamlines:



Isotherms:



$\longrightarrow$

δT

[De Vahl Davis 1983 Natural convection of air in a square cavity: A benchmark numerical solution. *Int. J. Num. Meth. Fluids*]

Main dimensionless control parameter: dimensionless measure of δT ?

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Idea: $R = \dfrac{\alpha \, \delta T \, \mathbf{g}}{\nu \Delta \mathbf{v}} = \dfrac{\alpha \, \delta T \, g \, d^2}{\nu V}$ with d the length scale of the cavity.

Determine V taking into account the feedback of **v** onto T. Where is this feedback ?

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with $\tau_{\text{therm}} = d^2/\kappa$ the heat diffusion time

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$$\Longrightarrow \boxed{R = \frac{\alpha\, \delta T\, g\, d^3}{\kappa \nu}}$$ **Rayleigh number** .

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Caution: $V = \kappa/d$ meaningful from the point of view of dimensional analysis - not always true regarding orders of magnitude !

Main dimensionless control parameter: Rayleigh number

$$R = \frac{\alpha\, \delta T\, g\, d^3}{\nu \kappa}$$

Order of magnitude for typical fluids ?

Fluid	T_0	thermal expansion coefficient α	kinematic viscosity ν	heat diffusivity κ
Water	$20\,^\circ\mathrm{C}$	$2\ 10^{-4}$ K^{-1}	$1\ 10^{-6}$ m^2/s	$1\ 10^{-7}$ m^2/s
Air	$27\,^\circ\mathrm{C}$		$2\ 10^{-5}$ m^2/s	$2\ 10^{-5}$ m^2/s

www.engineeringtoolbox.com

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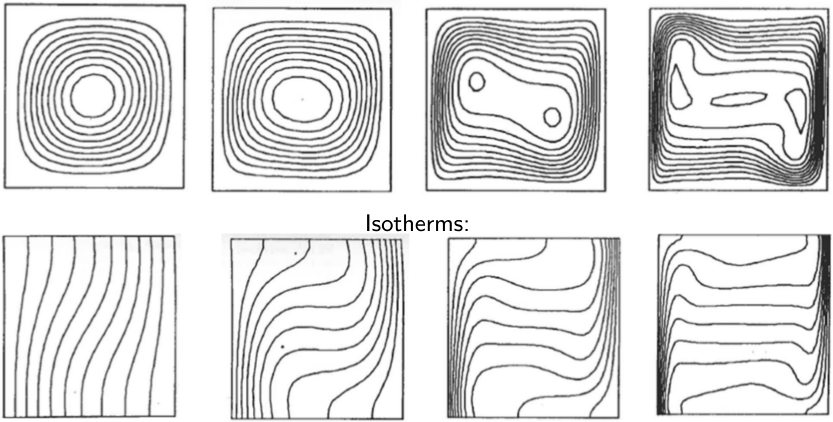
R large as soon as δT and d not too small !

www.engineeringtoolbox.com

Steady thermoconvection in a 2D differentially heated cavity

Streamlines:

Isotherms:

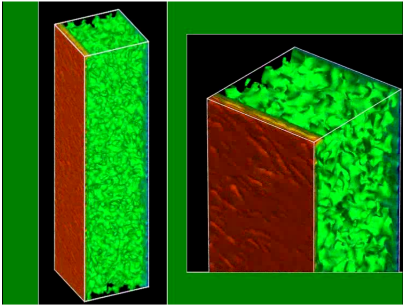


$R = \frac{\alpha \delta T g d^3}{\kappa \nu} = 10^3$ $R = 10^4$ $R = 10^5$ $R = 10^6$

[De Vahl Davis 1983 Natural convection of air in a square cavity: A benchmark numerical solution. *Int. J. Num. Meth. Fluids*]

Unsteady thermoconvection in a 3D differentially heated cavity

DNS at $R = 2\ 10^9$, for a height aspect ratio of 4 : initial condition isotherms:

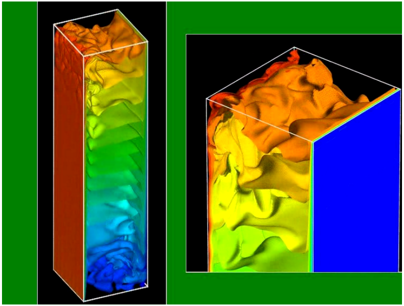


[Trias, Soria et al. 2007 DNS of 2 and 3-dimensional turbulent natural convection flows in a differentially heated cavity of aspect ratio 4. *J. Fluid Mech.*]

www.fxtrias.com/natural_convection.html

Unsteady thermoconvection in a 3D differentially heated cavity

DNS at $R = 2\ 10^9$, for a height aspect ratio of 4 : end-of-the-run isotherms:



[Trias, Soria et al. 2007 DNS of 2 and 3-dimensional turbulent natural convection flows in a differentially heated cavity of aspect ratio 4. *J. Fluid Mech.*]

The isotherms are, in the core of the cavity, roughly horizontal planes...

like in the high-R 2D case of De Vahl Davis, see frame 14 !..

The study of a **simpler 2D system** at $R = 10^6$

gives relevant informations for the **complex 3D system** at $R = 2\ 10^9$!

What we learnt about natural thermoconvection

- It is governed (in 1st approximation) by the **OB equations**:

$$\mathrm{div}\mathbf{v} = 0 \,, \quad \text{(MC)}$$

$$\frac{d\mathbf{v}}{dt} = -\alpha T\,\mathbf{g} - \nabla p'' + \nu\Delta\mathbf{v} \,, \quad \text{(NS)}$$

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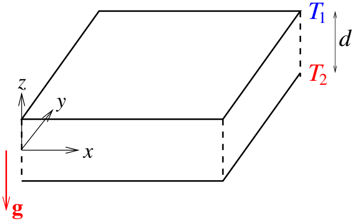
- ∇T not vertical $\Rightarrow$ **thermoconvection flows develop at once**.
- ∇T vertical $\Rightarrow$ **thermoconvection flows do not always start ?**
  how do they start ?

Study of the Rayleigh-Bénard system: plane cavity heated from below



The **OB equations**

$$\mathrm{div}\mathbf{v} = 0 \,, \qquad \text{(MC)}$$

$$\frac{d\mathbf{v}}{dt} = -\alpha T \, \mathbf{g} - \nabla p'' + \nu \Delta \mathbf{v} \,, \qquad \text{(NS)}$$

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always admit a **static solution**

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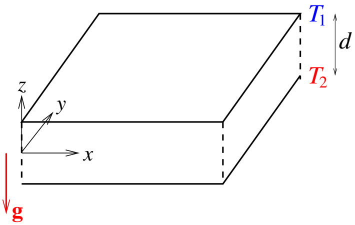

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always admit a **static solution** that satisfies the isothermal boundary conditions:

$$\mathbf{v} = \mathbf{0} \, , \quad T = T_0 - \delta T \frac{z}{d} \quad \text{with} \quad \delta T = T_2 - T_1 \, .$$

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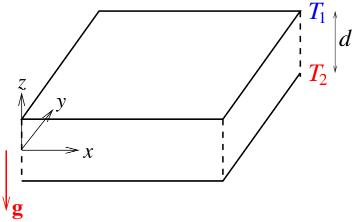

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Thus, how convection can set in ?

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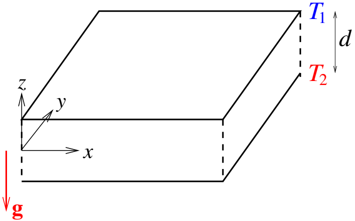

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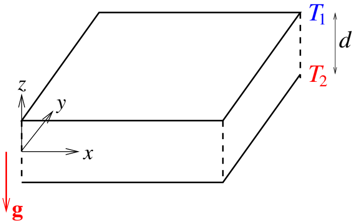
Thus, how convection can set in ? Through an instability of the static solution !

Study of the Rayleigh-Bénard system: dimensionless model

- Unit of length = thickness d
- Unit of time = heat diffus° time $\tau_{\text{therm}} = \dfrac{d^2}{\kappa}$
- Unit of velocity = $V = \dfrac{d}{\tau_{\text{therm}}} = \dfrac{\kappa}{d}$
- Unit of temperature = δT



Dimensionless time $t' = t/\tau_{\text{therm}} = t\kappa/d^2 =$

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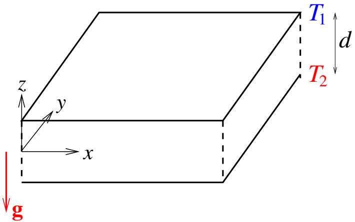

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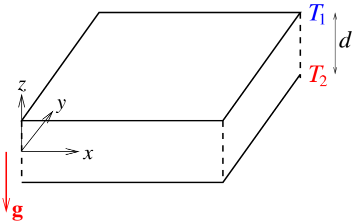

Dimensionless time $t' = t/\tau_{\text{therm}} = t\kappa/d^2$ = Fourier number

Introduce a dimensionless perturbat° of temperature θ, s.t. the dimensionless temperature

$$T' = T'_0 - z' + \theta$$

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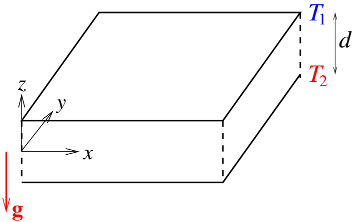

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⇒ dimensionless **OB equations**

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$$P^{-1}\frac{d\mathbf{v}}{dt} = R\theta\,\mathbf{e}_z - \nabla p + \Delta\mathbf{v} \,, \qquad \text{(NS)}$$

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with the **Rayleigh number** $R = \alpha\,\delta T\,g\,d^3/(\kappa\nu)$ and the **Prandtl number** $P = \nu/\kappa$.

Study of the Rayleigh-Bénard system: dimensionless model

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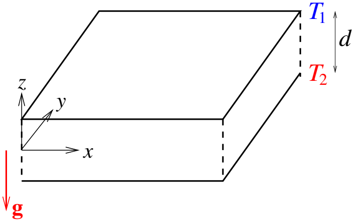

Isotropy of the problem in the horizontal plane $\Rightarrow$ focus on **2D** xz **solutions**

$$\mathbf{v} = v_x(x,z,t)\;\mathbf{e}_x + v_z(x,z,t)\;\mathbf{e}_z \;, \quad \theta = \theta(x,z,t) \;...$$

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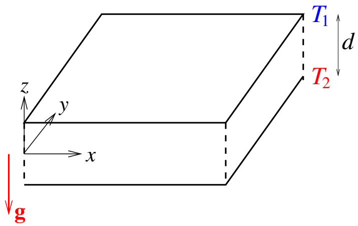

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How can one solve (MC) in a convenient manner ?

Study of the Rayleigh-Bénard system: dimensionless model

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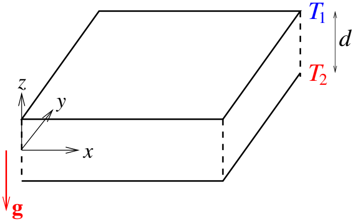

Isotropy of the problem in the horizontal plane $\Rightarrow$ focus on **2D** xz **solutions**

$$\mathbf{v} = v_x(x,z,t) \, \mathbf{e}_x + v_z(x,z,t) \, \mathbf{e}_z \, , \quad \theta = \theta(x,z,t) \, \ldots$$

How can one solve (MC) in a convenient manner ? Use a **streamfunction** ψ such that

$$\mathbf{v} = \mathbf{curl}(\psi \, \mathbf{e}_y) = (\nabla\psi) \times \mathbf{e}_y = -(\partial_z\psi) \, \mathbf{e}_x + (\partial_x\psi) \, \mathbf{e}_z \, .$$

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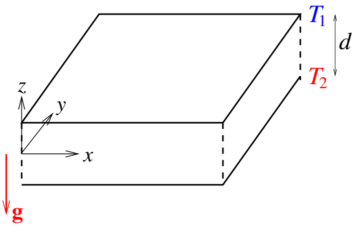

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How can one eliminate p in (NS) ?

Study of the Rayleigh-Bénard system: dimensionless model

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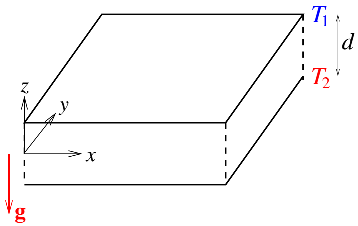

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How can one eliminate p in (NS) ? Consider $\mathbf{curl}(\mathrm{NS})\cdot\mathbf{e}_y$ i.e. the **vorticity equation**:

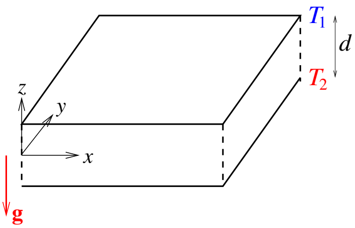
$$P^{-1}\partial_t(-\Delta\psi) + P^{-1}\left[\partial_z\left(\mathbf{v}\cdot\nabla v_x\right) - \partial_x\left(\mathbf{v}\cdot\nabla v_z\right)\right] = -R\partial_x\theta + \Delta(-\Delta\psi) . \qquad \text{(VortE)}$$

Study of 2D *xz* solutions of the Rayleigh-Bénard problem

Local state vector: $V = (\psi,\theta)$ s. t.

$$\mathbf{v} = -(\partial_z\psi)\,\mathbf{e}_x + (\partial_x\psi)\,\mathbf{e}_z \;,$$

$$T = T_0 - z + \theta \;,$$



obeys the system of coupled P.D.E.

$$\boxed{D\cdot\partial_t V = L_R\cdot V + N_2(V,V)}$$

with D, L_R linear, N_2 nonlinear **differential operators**. 1st eq. is the **vorticity equation**:

$$[D\cdot\partial_t V]_\psi = P^{-1}(-\Delta\partial_t\psi) \;,\quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi) \;,$$

$$[N_2(V,V)]_\psi = P^{-1}\left[\partial_x\left(\mathbf{v}\cdot\boldsymbol{\nabla}v_z\right) - \partial_z\left(\mathbf{v}\cdot\boldsymbol{\nabla}v_x\right)\right] \;,$$

2d eq. is the **heat equation**:

$$[D\cdot\partial_t V]_\theta = \partial_t\theta \;,\quad [L_R\cdot V]_\theta = \Delta\theta + v_z \;,\quad [N_2(V,V)]_\theta = -\mathbf{v}\cdot\boldsymbol{\nabla}\theta \;.$$

Study of 2D xz solutions of the Rayleigh-Bénard problem

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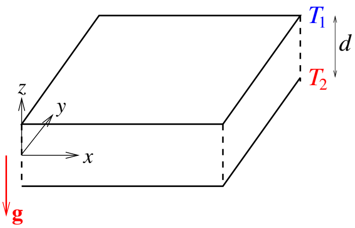

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What do we need to close this system ?

Study of 2D xz solutions of the Rayleigh-Bénard problem

Local state vector: $V = (\psi,\theta)$ s. t.

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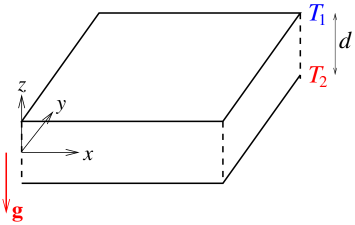

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What do we need to close this system ? Boundary conditions !

Study of 2D xz solutions of the Rayleigh-Bénard problem

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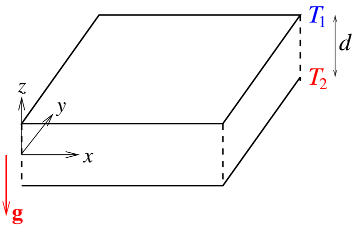

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Boundary conditions on θ :

Study of 2D xz solutions of the Rayleigh-Bénard problem

Local state vector: $V = (\psi, \theta)$ s. t.

$$\mathbf{v} = -(\partial_z \psi)\, \mathbf{e}_x + (\partial_x \psi)\, \mathbf{e}_z ,$$

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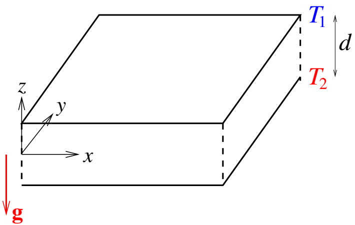

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Boundary conditions on θ : **isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

Study of 2D xz solutions of the Rayleigh-Bénard problem

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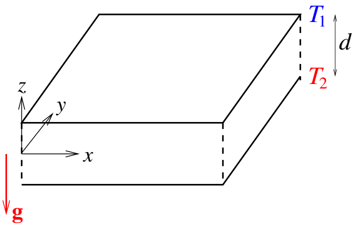

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$$[N_2(V,V)]_\psi = P^{-1}\left[\partial_z\left(\mathbf{v} \cdot \boldsymbol{\nabla} v_x\right) - \partial_x\left(\mathbf{v} \cdot \boldsymbol{\nabla} v_z\right)\right] , \qquad \text{(VortE)}$$

$$[D \cdot \partial_t V]_\theta = \partial_t \theta , \quad [L_R \cdot V]_\theta = \Delta \theta + v_z , \quad [N_2(V,V)]_\theta = -\mathbf{v} \cdot \boldsymbol{\nabla} \theta . \qquad \text{(HE)}$$

Boundary conditions on θ : **isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

Boundary conditions on ψ i.e. $\mathbf{v}$:

Study of 2D *xz* solutions of the Rayleigh-Bénard problem

Local state vector: $V = (\psi,\theta)$ s. t.

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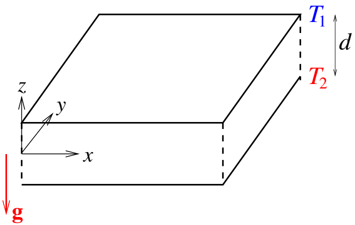

$$[D\cdot\partial_t V]_\psi = P^{-1}(-\Delta\partial_t\psi) , \quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi) , \qquad \text{(VortE)}$$

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Boundary conditions on θ : **isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

Boundary conditions on ψ i.e. $\mathbf{v}$:

no-slip boundaries: $\mathbf{v} = \mathbf{0}$ i.e. $\partial_x\psi = \partial_z\psi = 0$ if $z = \pm 1/2$

Study of 2D *xz* solutions of the Rayleigh-Bénard problem

Local state vector: $V \;=\; (\psi,\theta)$ s. t.

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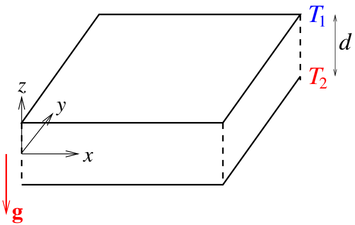
$$\boxed{D\cdot\partial_t V \;=\; L_R\cdot V \;+\; N_2(V,V)}\;,$$



$$[D\cdot\partial_t V]_\psi \;=\; P^{-1}(-\Delta\partial_t\psi)\;,\quad [L_R\cdot V]_\psi \;=\; -R\partial_x\theta \;+\; \Delta(-\Delta\psi)\;, \qquad \text{(VortE)}$$

$$[N_2(V,V)]_\psi \;=\; P^{-1}\big[\partial_z\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_x\big) - \partial_x\big(\mathbf{v}\cdot\boldsymbol{\nabla} v_z\big)\big]\;, \qquad \text{(VortE)}$$

$$[D\cdot\partial_t V]_\theta \;=\; \partial_t\theta\;,\quad [L_R\cdot V]_\theta \;=\; \Delta\theta \;+\; v_z\;,\quad [N_2(V,V)]_\theta \;=\; -\mathbf{v}\cdot\boldsymbol{\nabla}\theta\;. \qquad \text{(HE)}$$

Boundary conditions on θ : **isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

Boundary conditions on ψ i.e. $\mathbf{v}$:

no-slip boundaries: $\mathbf{v} = \mathbf{0}$ i.e. $\partial_x\psi = \partial_z\psi = 0$ if $z = \pm 1/2$

or **slip boundaries:** $v_z = 0$ i.e. $\partial_x\psi = 0$ if $z = \pm 1/2$

Study of 2D xz solutions of the Rayleigh-Bénard problem

Local state vector: $V \;=\; (\psi,\theta)$ s. t.

$$\mathbf{v} \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z \;,$$

$$T \;=\; T_0 \;-\; z \;+\; \theta \;,$$

obeys the system of coupled P.D.E.

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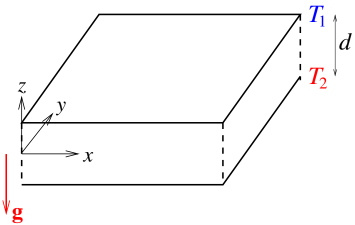

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or **slip boundaries:** $v_z = 0$ i.e. $\partial_x\psi = 0$ if $z = \pm 1/2$ **without shear stress !**

Shear stresses or tangential stresses

Come only from **viscous stresses**.

In physical units, the **viscous stress vector**

$$\mathbf{T} \;=\; \boldsymbol{\tau}\cdot\mathbf{n}$$

with the **viscous stress tensor**

$$\boldsymbol{\tau} \;=\; 2\eta\mathbf{S}$$

and the **rate-of-strain tensor**

$$\mathbf{S} \;=\; \frac{1}{2}\Big(\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\nabla}\mathbf{v}^T\Big)\;,$$

$\mathbf{n}$ the unit vector normal to the boundary, pointing outward.

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$\mathbf{n}$ the unit vector normal to the boundary, pointing outward.

Here

$$\mathbf{v} \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z$$

and

$$\mathbf{n} \;=\; \mp\,\mathbf{e}_z$$

$$\Longrightarrow\quad \text{Shear stress } T_x \;=\; \mp\,\eta(\partial_z v_x + \partial_x v_z) \quad \text{if} \quad z = \pm 1/2\;.$$

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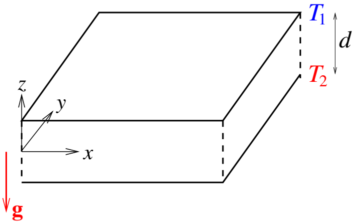
$$\Longrightarrow \quad \text{Shear stress } T_x \;=\; \mp\,\eta(\partial_z v_x + \partial_x v_z) \quad \text{if} \quad z = \pm 1/2\;.$$

Since $v_z = 0$, $\quad T_x \;=\; 0$ at the boundaries $\quad\Longleftrightarrow\quad \partial_z v_x = 0 \quad \text{if} \quad z = \pm 1/2\;.$

Study of 2D *xz* solutions of the Rayleigh-Bénard problem

Local state vector: $V \;=\; (\psi,\theta)$ s. t.



$$\mathbf{v} \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z \;,$$

$$T \;=\; T_0 \;-\; z \;+\; \theta \;,$$

$$\boxed{D\cdot\partial_t V \;=\; L_R\cdot V \;+\; N_2(V,V)} \;,$$

$$[D\cdot\partial_t V]_\psi \;=\; P^{-1}(-\Delta\partial_t\psi)\;,\quad [L_R\cdot V]_\psi \;=\; -R\partial_x\theta \;+\; \Delta(-\Delta\psi)\;, \qquad \text{(VortE)}$$

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Boundary conditions on θ : **Isothermal boundaries:** $\theta = 0$ if $z = \pm 1/2$.

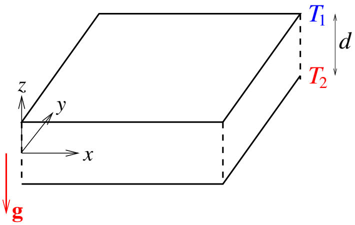
Boundary conditions on ψ i.e. $\mathbf{v}$:

Slip boundaries: $v_z = 0$ and $\partial_z v_x = 0 \iff \partial_x\psi = \partial_z^2\psi = 0$ if $z = \pm 1/2$.

Extended geometry in the xy **plane:** no B.C. or periodic B.C. under $x \mapsto x + L$.

Linear stability analysis of the 2D xz Rayleigh-Bénard problem

Local state vector: $V = (\psi,\theta)$ s. t.



$$\mathbf{v} = -(\partial_z\psi)\,\mathbf{e}_x + (\partial_x\psi)\,\mathbf{e}_z \ ,$$

$$T = T_0 - z + \theta \ ,$$

$$\boxed{D\cdot\partial_t V = L_R\cdot V} \ ,$$

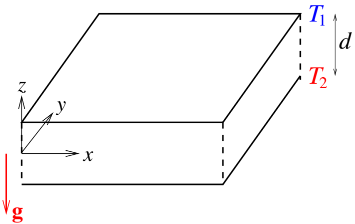
$$[D\cdot\partial_t V]_\psi = P^{-1}(-\Delta\partial_t\psi) \ , \quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi) \ , \qquad \text{(VortE)}$$

$$[D\cdot\partial_t V]_\theta = \partial_t\theta \ , \quad [L_R\cdot V]_\theta = \Delta\theta + v_z \ , \qquad \text{(HE)}$$

$$\partial_x\psi = \partial_z^2\psi = \theta = 0 \quad \text{if} \quad z = \pm 1/2 \ .$$

Linear stability analysis of the 2D *xz* Rayleigh-Bénard problem

Local state vector: $V = (\psi,\theta)$ s. t.



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$$\partial_x\psi = \partial_z^2\psi = \theta = 0 \quad \text{if} \quad z = \pm 1/2 \ .$$

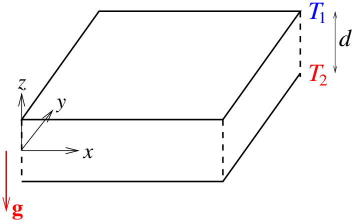
Ex. 1.1: Normal mode analysis: the solution of the initial value problem is the superposition of normal modes that are Fourier modes in $\exp(ikx)$,

$$V = V_1(k,N)\ \exp[\sigma(k,N)\ t] \quad \text{with} \quad V_1(k,N) = (\widehat{\Psi}(z), \widehat{\Theta}(z))\ \exp(ikx) \ ,$$

k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes, $\sigma(k,N)$ the **temporal eigenvalue**.

Linear stability analysis of the 2D *xz* Rayleigh-Bénard problem

Local state vector: $V \;=\; (\psi,\theta)$ s. t.



$$\mathbf{v} \;=\; -(\partial_z\psi)\,\mathbf{e}_x \;+\; (\partial_x\psi)\,\mathbf{e}_z \;,$$

$$T \;=\; T_0 \;-\; z \;+\; \theta \;,$$

$$\boxed{D\cdot\partial_t V \;=\; L_R\cdot V} \;,$$

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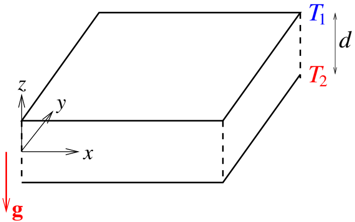
k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes, $\sigma(k,N)$ the **temporal eigenvalue**.

Boundary conditions: $\widehat{\Psi} = \widehat{\Psi}'' = 0 \;,\quad \widehat{\Theta} = 0 \quad \text{if} \quad z = \pm 1/2 \;.$

Normal mode analysis of the 2D xz Rayleigh-Bénard problem



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$$T = T_0 - z + \theta \;,$$

$$\boxed{\sigma D\cdot V = L_R\cdot V} \;,$$

$$[D\cdot V]_\psi = P^{-1}(-\Delta\psi)\;,\quad [L_R\cdot V]_\psi = -R\partial_x\theta + \Delta(-\Delta\psi)\;, \qquad \text{(VortE)}$$

$$[D\cdot V]_\theta = \theta\;,\quad [L_R\cdot V]_\theta = \Delta\theta + v_z\;, \qquad \text{(HE)}$$

$$\partial_x\psi = \partial_z^2\psi = \theta = 0 \quad \text{if} \quad z = \pm 1/2 \;.$$

Ex. 1.1: Generalized eigenvalue problem solved by normal modes analysis: most relevant normal modes are Fourier modes in $\exp(ikx)$ and have a z-profile in $\cos(\pi z)$,

$$V = V_1(k,N)\,\exp[\sigma(k,N)\,t] \quad \text{with} \quad V_1(k,N) = (\Psi,\,\Theta)\,\exp(ikx)\,\cos(\pi z)\;,$$

k = **horizontal wavenumber**, $k \neq 0$, N another label to mark normal modes,

$\sigma(k,N)$ the **temporal eigenvalue**.

They satisfy the boundary conditions: $\widehat{\Psi} = \widehat{\Psi}'' = 0\;,\quad \widehat{\Theta} = 0 \quad \text{if} \quad z = \pm 1/2\;.$