Example of a technical report: An introduction to the RANS approach for Turbulence modelling

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1 Introduction

This is an example of a technical report that you can download on the web page

`http://emmanuelplaut.perso.univ-lorraine.fr/latex` .

I give you some extracts of Plaut *et al.* (2021), which corresponds to a module that I teach at the ***École Nationale Supérieure des Mines de Nancy*** (Mines Nancy). The logo of Mines Nancy is presented on the figure 1. My part in Plaut *et al.* (2021) is devoted to the ***Reynolds Averaged Navier-Stokes*** (RANS) ***approach*** for ***Turbulence modelling***.

2 The basic concepts and equations of fluid dynamics - Notations

2.1 Cartesian coordinates, tensors and differential operators

We most often use a ***cartesian system of coordinates*** of origin O, associated to the laboratory frame. The coordinates are denoted (x, y, z) or (x_1, x_2, x_3), and we use Einstein's convention of summation over repeated indices, e.g. the position vector is

$$\mathbf{x} \;=\; x_i \mathbf{e}_i$$

with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the orthonormal base vectors. We do not use overbars to designate vectors, since the overbars will be used to define various averages, e.g. in section 3, 'Reynolds averages'. Vectors and tensors of order 2 will thus be simply denoted by boldface characters. The use of tensorial intrinsic

Fig. 1 : The logo of the Engineering School where I teach. Note that the Figure number appears in bold font and that the caption appears with a small font.

notations will be minimized, to save some energy to face other difficulties.
Last but not least, ∂_t (resp. ∂_{x_i}) denotes the differential operator that takes the partial derivative with respect to the time t (resp. coordinate x_i).

2.2 Dynamics of incompressible newtonian fluids

The ***eulerian velocity field*** $\mathbf{v}(\mathbf{x}, t)$ is used to describe the flow of ***incompressible newtonian fluids***. The ***incompressibility*** means that the ***mass density*** ρ is uniform and constant. Therefore the ***mass conservation equation*** reads

$$\mathrm{div}\mathbf{v} \;=\; 0 \quad\Longleftrightarrow\quad \boxed{\partial_{x_i} v_i \;=\; 0}\;, \tag{1}$$

which means that the velocity field $\mathbf{v}$ is conservative.
The ***Cauchy stress tensor*** $\boldsymbol{\sigma}$ determines the surface force $d^2\mathbf{f}$ exerted on a small surface of area d^2A and normal unit vector $\mathbf{n}$ pointing outwards, by the exterior onto the interior, through

$$d^2\mathbf{f} \;=\; \boldsymbol{\sigma}\cdot\mathbf{n}\; d^2A \;=\; \sigma_{ij} n_j\; d^2A\;\mathbf{e}_i\;. \tag{2}$$

The stresses σ_{ij} are given by the sum of ***pressure*** and ***viscous stresses***,

$$\sigma_{ij} \;=\; -\,p_{\text{static}}\delta_{ij} \;+\; 2\eta S_{ij}(\mathbf{v}) \tag{3}$$

with p_{static} the ***static pressure***, η the ***dynamic viscosity*** of the fluid, $\mathbf{S}(\mathbf{v})$ the ***rate-of-strain tensor*** defined as the symmetric part of the velocity gradient, i.e., in components,

$$\boxed{S_{ij}(\mathbf{v}) \;:=\; \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)}\;. \tag{4}$$

In this equation, the sign $:=$ means a definition.
The linear momentum equation is the ***Navier-Stokes equation***

$$\rho[\partial_t v_i \;+\; \partial_{x_j}(v_i v_j)] \;=\; \rho g_i \;+\; \partial_{x_j}\sigma_{ij} \;=\; \rho g_i - \partial_{x_i} p_{\text{static}} \;+\; \partial_{x_j}(2\eta S_{ij}(\mathbf{v}))\;, \tag{5}$$

where g_i are the components of the acceleration due to gravity. Usually, a ***modified pressure*** that includes a gravity term,

$$p \;=\; p_{\text{static}} \;+\; \rho g Z\;, \tag{6}$$

where Z is a vertical coordinate, is used, to group the first two terms on the r.h.s. of equation (5), which reads therefore

$$\boxed{\rho[\partial_t v_i \;+\; \partial_{x_j}(v_i v_j)] \;=\; -\,\partial_{x_i} p \;+\; \partial_{x_j}(2\eta S_{ij}(\mathbf{v}))}\;. \tag{7}$$

We will most often use the modified pressure p instead of the static pressure p_{static} .
After dividing by the mass density, we get another form of the ***Navier-Stokes equation***,

$$\boxed{\partial_t v_i \;+\; v_j\partial_{x_j} v_i \;=\; -\,\frac{1}{\rho}\partial_{x_i} p \;+\; \partial_{x_j}(2\nu S_{ij}(\mathbf{v}))} \tag{8}$$

with $\nu = \eta/\rho$ the ***kinematic viscosity*** of the fluid. We have used the mass conservation equation (1), $\partial_{x_j} v_j = 0$, to rewrite the nonlinear term on the l.h.s.; this whole l.h.s. is in fact the acceleration of the fluid particle (in the sense of the continuum mechanics) that passes through $\mathbf{x}$ at time t...

3 Reynolds decomposition, equations and stresses

3.1 Reynolds average and decomposition

A natural and most important approach in turbulence is the ***statistical approach***. A turbulent flow experiment (possibly, a numerical experiment !) may be repeated N times, with N a large integer, and one may average for instance the pressure field $p(\mathbf{x},t)$ over all the realizations. Denoting $p^n(\mathbf{x},t)$ the pressure field at position $\mathbf{x}$ and time t during the n^{th} realization, the ***ensemble average*** or ***Reynolds average*** of the pressure is

$$\boxed{\overline{p}(\mathbf{x},t) \;=\; \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} p^n(\mathbf{x},t)} \;. \tag{9}$$

Often, the turbulence is ***'stationary'***, and one may, with a reasonable hypothesis of ***ergodicity***, use alternately a ***time average*** to define the ***mean pressure***

$$\overline{p}(\mathbf{x}) \;=\; \langle p(\mathbf{x},t)\rangle_t \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T p(\mathbf{x},t)\;dt \tag{10}$$

where, on the r.h.s., in the integral, it might be relevant to write p^1 instead of p, since we refer to one realization (say, the first one, labelled $n=1$) of the flow.
The ***Reynolds decomposition*** splits the pressure p and the velocity fields $\mathbf{v}$
into ***mean values*** + ***fluctuations***, according to

$$\begin{array}{lccccc} & \boxed{p \;=\; P \;+\; p'} \;, & \boxed{v_i \;=\; U_i \;+\; u_i} & \Longleftrightarrow & \boxed{\mathbf{v} \;=\; \mathbf{V} \;+\; \mathbf{u}} & \\ \text{with} & P \;=\; \overline{p} \;, & U_i \;=\; \overline{v_i} & \Longleftrightarrow & \mathbf{V} \;=\; \overline{\mathbf{v}}\;, & \\ \text{hence} & \overline{p'} = 0 \;, & \overline{u_i} \;=\; 0 & \Longleftrightarrow & \overline{\mathbf{u}} \;=\; \mathbf{0}\;. & \end{array} \tag{11}$$

3.2 Reynolds equations and stresses or second moments

As stated in the introduction, the fluid is assumed to be ***incompressible***, hence the ***mass conservation equation*** is given by (1),

$$\partial_{x_i} v_i \;=\; 0\;. \tag{12}$$

From the definition (9), it is clear the one has the ***commutation rules***

$$\overline{\partial_t p} \;=\; \partial_t\overline{p} \quad \text{and} \quad \forall i\;,\quad \overline{\partial_{x_i} p} \;=\; \partial_{x_i}\overline{p}\;. \tag{13}$$

This applies to any real-valued field, i.e. also to the velocity components. Therefore, by taking the Reynolds average of the equation (12), we obtain

$$\overline{\partial_{x_i} v_i} \;=\; \partial_{x_i}\overline{v_i} \;=\; 0 \quad\Longleftrightarrow\quad \boxed{\partial_{x_i} U_i \;=\; 0}\;, \tag{14}$$

i.e. the mean flow is also conservative. This equation (14) is the ***first RANS equation***. Since the mass conservation equation (12) is linear, its average form (14) is exactly the same.

Let us now consider the ***Navier-Stokes equation*** (7),

$$\rho[\partial_t v_i \;+\; \partial_{x_j}(v_i v_j)] \;=\; -\,\partial_{x_i} p \;+\; \partial_{x_j}(2\eta S_{ij}(\mathbf{v}))\;. \tag{15}$$

On purpose, the nonlinear advection term on the l.h.s. has been written under a 'conservative' form. When one takes the Reynolds average of the equation (15), the averages of the linear terms yield no surprise according to the commutation rules (13), i.e. we get

$$\rho\big(\partial_t U_i \;+\; \partial_{x_j}\overline{v_i v_j}\;\big) \;=\; -\,\partial_{x_i} P \;+\; \partial_{x_j}(2\eta S_{ij}(\mathbf{V}))\;. \tag{16}$$

On the contrary, the nonlinear term has to be treated carefully. Indeed

$$\overline{v_i v_j} \;=\; \overline{(U_i+u_i)(U_j+u_j)} \;=\; \overline{U_i U_j} \;+\; \overline{U_i u_j} \;+\; \overline{u_i U_j} \;+\; \overline{u_i u_j} \;=\; U_i U_j \;+\; U_i \overline{u_j} \;+\; \overline{u_i} U_j \;+\; \overline{u_i u_j} \;.$$

To transform the second and third terms on the r.h.s., we made use of the fact that $V = U_i$ or U_j do not depend on the realization, hence, with $w = u_j$ or u_i ,

$$\overline{Vw} \;=\; \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} V(\mathbf{x},t)\; w^n(\mathbf{x},t) \;=\; V(\mathbf{x},t) \;\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w^n(\mathbf{x},t) \;=\; V(\mathbf{x},t)\; \overline{w}(\mathbf{x},t) \;=\; V\; \overline{w} \;.$$

Since $\mathbf{u}$ is a fluctuating vector field of zero average by its definition (11), we get

$$\overline{v_i v_j} \;=\; U_i U_j \;+\; \overline{u_i u_j} \quad\Longleftrightarrow\quad \overline{u_i u_j} \;=\; \overline{v_i v_j} \;-\; \overline{v_i}\;\overline{v_j} \;. \tag{17}$$

Using the terminology of statistics, the first of the equations (17) reads

$$\mathrm{mean}(v_i v_j) \;=\; (\mathrm{mean}(v_i))\;(\mathrm{mean}(v_j)) \;+\; \mathrm{covariance}(v_i v_j) \;. \tag{18}$$

Most importantly, if the fluctuations u_i and u_j are correlated, the ***second moment***

$$\boxed{\overline{u_i u_j} \;=\; \mathrm{covariance}(v_i v_j)} \tag{19}$$

does not vanish. Thus we get a 'fluctuation-induced' contribution in the ***RANS equation*** (16), that is put on the r.h.s.,

$$\boxed{\rho[\partial_t U_i \;+\; \partial_{x_j}(U_i U_j)] \;=\; -\,\partial_{x_i} P \;+\; \partial_{x_j}(2\eta S_{ij}(\mathbf{V})) \;+\; \partial_{x_j}\tau_{ij}} \tag{20}$$

with the ***Reynolds stresses***

$$\boxed{\tau_{ij} \;=\; -\,\rho\overline{u_i u_j} \;=\; -\,\rho\;\mathrm{covariance}(v_i v_j)} \;. \tag{21}$$

The equation (20) is the ***second RANS equation***, or the ***RANS momentum equation***. It shows that, if the fluctuations u_i and u_j are correlated, they feedback onto the mean-flow through the term τ_{ij} that corrects the viscous stress $2\eta S_{ij}(\mathbf{V})$ (see the equation 3) in the evolution equation for U_i ; this 'analogous role' explains the terms 'Reynolds stress' or 'Reynolds stresses', though there are no surface forces (see the equation 2) behind this.

It is often convenient to divide the second RANS equation (20) by the mass density, to obtain an equation for the 'mean acceleration'

$$\boxed{\frac{DU_i}{Dt} \;:=\; \partial_t U_i \;+\; U_j \partial_{x_j} U_i \;=\; -\,\frac{1}{\rho}\partial_{x_i} P \;+\; \partial_{x_j}(2\nu S_{ij}(\mathbf{V})) \;-\; \partial_{x_j}\overline{u_i u_j}} \;. \tag{22}$$

We face a ***closure problem***: to close the system of the Reynolds equations (14) and (20), or, equivalently, of the equations (14) and (22), to be able to determine the first moments $P = \overline{p}$ and $U_i = \overline{v_i}$, we have to model the second moments

$$\overline{u_i u_j} \;.$$

A solution to this closure problem is a ***RANS model*** !..

References

Plaut, E., Peinke, J. & Hölling, M. 2021 *Turbulence & Wind Energy*. Cours de Mines Nancy (3A), http://emmanuelplaut.perso.univ-lorraine.fr/twe/pol.pdf.