# Advanced Fluid Mechanics

## Transition to turbulence & turbulence

## Applications to thermoconvection, aerodynamics & wind energy

### General planning:

| Sess° - Man | Date | Content |
|---|---|---|
| 1 - **EP** | 18/12/17 | **RB*** Thermoconvection: linear stability analysis |
| 2 - **EP** | 08/01/18 | **RB** Thermoconvection: weakly nonlinear stability analysis |
| 3 - **EP** | 12/01 | **RB** Thermoconvection: nonlinear phenomena |
| 4 - **EP** | 19/01 | Aerodynamics of **OSF***: linear stability analysis |
| → 5 - **EP** | 22/01 | Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses |
| 6 - **EP** | 24/01 | Aerodynamics of **OSF**: nonlinear phenomena |
| 7 - **JP** | 06/02 | Wind resources - Conversion principles - Aero |
| 8 - **JP** | 07/02 | Aero - Wind field and Turbulence |
| 9 - **JP** | 08/02 | Wind field and Turbulence - Conversion dynamics - Stochastic proc. |
| **JP** | 08/02 | **General conf.: Turbulence & Wind Energy Research** |
| **EP** | 12/02 | **Examination** |

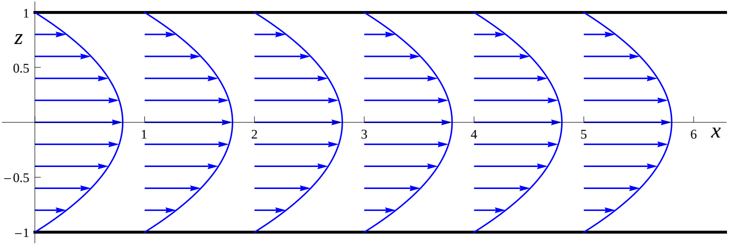
**RB*** = Rayleigh-Bénard **OSF*** = open shear flows

### Today: session 5: transition in open shear flows:

- Numerical linear stability analysis of plane Poiseuille flow (PPF) : TS waves
- Setup of the weakly nonlinear analysis of TS waves in PPF...

# When and how 2D $xz$ laminar open shear flows get unstable ?

# Example: plane Poiseuille flow




Viscous flow between two plates at $z = \pm h$ :

$$\bar{\mathbf{v}} = U(z)\,\bar{\mathbf{e}}_x = U_0(1-(z/h)^2)\,, \quad \widehat{p} = p + \rho g Z = -Gx \quad \text{with} \quad G = 2\eta\frac{U_0}{h^2}\,.$$

Particular case of **plane parallel flow** !

# When and how 2D $xz$ laminar open shear flows get unstable ?

## General example: plane parallel flows

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 = U(z)\, \overline{\mathbf{e}}_x\ , \quad \widehat{p} = p + \rho g Z = 0 \ \text{ in an inviscid fluid,}$$

$$\widehat{p} = p + \rho g Z = -Gx \ \text{ in a viscous fluid,}$$

is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

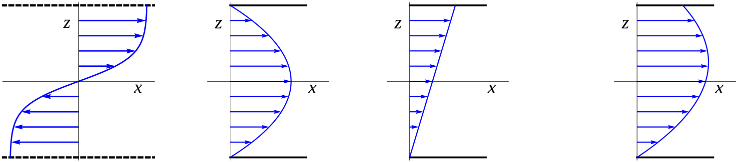
$$\rho\big[\partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \overline{\boldsymbol{\nabla}})\overline{\mathbf{v}}\big] = -\overline{\boldsymbol{\nabla}}\widehat{p} + \eta \overline{\boldsymbol{\Delta}}\overline{\mathbf{v}}$$

$$\Longleftrightarrow \qquad \overline{\mathbf{0}} = G\overline{\mathbf{e}}_x + \eta U''(z)\overline{\mathbf{e}}_x$$

whatever $U(z)$ in an inviscid fluid,

provided $U(z) = \alpha + \beta z + \gamma z^2$ in a viscous fluid.

Mixing layer | Poiseuille flow | Couette flow | Couette-Poiseuille flow




inviscid fl. | viscous fl. | viscous fl. | viscous fl.

# Stability analysis of plane parallel flows

Basic flow:

$$\overline{\mathbf{v}}_0 = U(z)\,\overline{\mathbf{e}}_x\,, \quad \widehat{p}_0 = p_0 + \rho g Z = -Gx \quad \text{with} \quad G = 0 \text{ in an inviscid fluid,}$$

$$G > 0 \text{ in a viscous fluid.}$$

Basic flow with **perturbations**:

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \overline{\mathbf{u}}\,, \quad \widehat{p} = \widehat{p}_0 + p'$$

$$\partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \overline{\boldsymbol{\nabla}})\overline{\mathbf{v}} = -(1/\rho)\overline{\boldsymbol{\nabla}}\widehat{p} + \nu \overline{\boldsymbol{\Delta}}\overline{\mathbf{v}} \qquad \text{(NS)}$$

$$\partial_t \overline{\mathbf{u}} + U' u_z \overline{\mathbf{e}}_x + U\partial_x \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}})\overline{\mathbf{u}} = -(1/\rho)\overline{\boldsymbol{\nabla}} p' + \nu \overline{\boldsymbol{\Delta}}\overline{\mathbf{u}} \qquad \text{(NS)}$$

$$\mathrm{div}\overline{\mathbf{v}} = \mathrm{div}\overline{\mathbf{u}} = 0 \qquad \text{(MC)}$$

▷ Unit of length = $h$ half-width of the channel, thickness of the mixing layer...
▷ Unit of velocity = $U_0 = \max_z U(z)$ scale of $U$
▷ Unit of time = $h/U_0$ advection time

$$\partial_t \overline{\mathbf{u}} + U' u_z \overline{\mathbf{e}}_x + U\partial_x \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}})\overline{\mathbf{u}} = -\overline{\boldsymbol{\nabla}} p'' + R^{-1}\,\overline{\boldsymbol{\Delta}}\overline{\mathbf{u}} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R = U_0 h/\nu}$ , $R = \infty$ in an inviscid fluid.

# 2D $xz$ stability analysis of plane parallel flows

Dimensionless equations for the **perturbations** $\overline{\mathbf{u}}$ of velocity and $p''$ of pressure:

$$\partial_t \overline{\mathbf{u}} \; + \; U' u_z \overline{\mathbf{e}}_x \; + \; U \partial_x \overline{\mathbf{u}} \; + \; (\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}}) \overline{\mathbf{u}} \; = \; - \overline{\boldsymbol{\nabla}} p'' \; + \; R^{-1} \, \overline{\boldsymbol{\Delta}} \overline{\mathbf{u}} \, , \qquad \text{(NS)}$$

$$\mathrm{div} \overline{\mathbf{u}} \; = \; 0 \, . \qquad \text{(MC)}$$

**2D** $xz$ **perturbations** can be defined by their **streamfunction** $\psi(x,z)$ :

$$\overline{\mathbf{u}} \; = \; \overline{\mathbf{curl}}(\psi \; \overline{\mathbf{e}}_y) \; = \; (\overline{\boldsymbol{\nabla}} \psi) \times \overline{\mathbf{e}}_y \; = \; - (\partial_z \psi) \; \overline{\mathbf{e}}_x \; + \; (\partial_x \psi) \; \overline{\mathbf{e}}_z \; .$$

We can eliminate $p''$ in (NS) by considering $\overline{\mathbf{curl}}(\mathrm{NS}) \cdot \overline{\mathbf{e}}_y$ i.e. the **vorticity equation**:

$$\partial_t(-\Delta\psi) + \left[\partial_z\left(\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_x\right) - \partial_x\left(\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_z\right)\right] \; = \; R^{-1} \Delta(-\Delta\psi) + U \partial_x(\Delta\psi) - U''(\partial_x \psi) \quad \text{(Vort)}$$

$$\iff \quad \boxed{D \cdot \partial_t \psi \; = \; L_R \cdot \psi \; + \; N_2(\psi,\psi)} \; . \qquad \text{(Vort)}$$

Boundary conditions:

viscous fluid : $\overline{\mathbf{u}} = \overline{\mathbf{0}} \iff \partial_x \psi \; = \; \partial_z \psi \; = \; 0$ if $z = z_\pm$ ,

inviscid fluid : $u_z = 0 \iff \partial_x \psi \; = \; 0$ if $z = z_\pm$ .

# 2D $xz$ linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \; = \; L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \; = \; -\Delta \partial_t \psi \; , \quad L_R \cdot \psi \; = \; R^{-1} \Delta(-\Delta \psi) \; + \; U \partial_x (\Delta \psi) - U''(\partial_x \psi) \; ,$$

$$\begin{array}{lccccl} \text{viscous fluid :} & \bar{\mathbf{u}} = \bar{\mathbf{0}} & \Longleftrightarrow & \partial_x \psi \; = \; \partial_z \psi \; = \; 0 & \text{if} & z = z_\pm \; , \\ \text{inviscid fluid :} & u_z = 0 & \Longleftrightarrow & \partial_x \psi \; = \; 0 & \text{if} & z = z_\pm \; . \end{array}$$

**Normal mode analysis:**

$$\psi \; = \; \Psi_n(z) \; \exp(ikx + \sigma t) \; = \; \Psi_n(z) \; \exp[ik(x - c_r t)] \; \exp(kc_i t)$$

with $k \; =$ **horizontal wavenumber**, $k \neq 0$, $n$ another label to mark normal modes,
$\sigma \; =$ **temporal eigenvalue**.

Most often the bulk velocity of the basic flow $\langle U \rangle_z > 0 \quad \Rightarrow$ by advection

$\sigma \; = \; -i\omega \; = \; -ikc$ with $c$ the **complex phase velocity**,
$c_r > 0$ the **real phase velocity**,
$kc_i > 0$ (resp. $< 0$) the **growth rate** (resp. damping rate).

# 2D $xz$ linear stability analysis of plane parallel flows

$$-\sigma\Delta\psi = R^{-1}\Delta(-\Delta\psi) + U\partial_x(\Delta\psi) - U''(\partial_x\psi) \qquad \text{(Vort)}$$

$$\Longleftrightarrow \quad ikc\Delta\psi = R^{-1}\Delta(-\Delta\psi) + ikU\Delta\psi - ikU''\psi \qquad \text{(Vort)}$$

$$\Longleftrightarrow \quad \boxed{(U-c)\Delta\psi - U''\psi = (ikR)^{-1}\Delta\Delta\psi} \qquad \text{(Vort)}$$

**Orr - Sommerfeld eq.** in a viscous fluid, **Rayleigh eq.** in an inviscid fluid ($R = \infty$)

BC at $z = z_\pm$ : viscous fluid: $\psi = \partial_z\psi = 0$ ; inviscid fluid: $\psi = 0$ .

**Normal mode analysis:**

$$\psi = \Psi_n(z)\ \exp(ikx + \sigma t) = \Psi_n(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

with $k$ = **horizontal wavenumber**, $k \neq 0$, $n$ another label to mark normal modes,
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$kc_i > 0$ (resp. $< 0$) the **growth rate** (resp. damping rate).

# 2D $xz$ linear stability analysis of inviscid plane parallel flows

**Normal mode analysis:** assume there is at least one **amplified mode**

$$\psi = \Psi(z)\ \exp(ikx - ikct) = \Psi(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

with $c_r$ the **real phase velocity**, $kc_i > 0$ the **growth rate**.

It satisfies the **Rayleigh equation** $\boxed{(U - c)\Delta\psi - U''\psi = 0}$

with the BC $\psi = 0$ if $z = z_\pm$.

**Exercise 2.1 Rayleigh's inflection point criterion**

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, $k$ and $c$.

▷ By multiplication with a suitable function and integration over $z \in [z_-, z_+]$, show that

$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz + \int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{U(z) - c} dz = 0$$

and

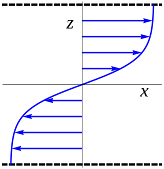
$$\int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{|U(z) - c|^2} dz = 0 \Rightarrow \text{ if } U'' \neq 0\ ,\ U'' \text{ must change sign somewhere,}$$

there must exist an **inflection point** in the $U$-profile.

# Instability of an inviscid plane parallel flow, the mixing layer

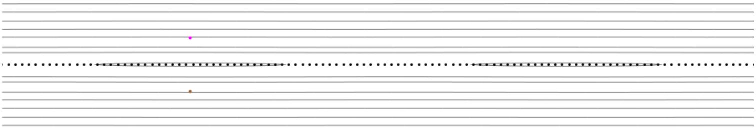
The hyperbolic tangent **mixing layer**

$$\overline{\mathbf{v}}_0 = U_0 \tanh(z/h)\ \overline{\mathbf{e}}_x$$




displays a **Kelvin-Helmholtz Instability** !

Initial condition $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \overline{\mathbf{u}}$ with $\overline{\mathbf{u}}$ small:

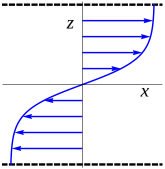


[ Plaut E. *Mécanique des fluides.* Cours Mines Nancy 2A ]

# Instability of an inviscid plane parallel flow, the mixing layer

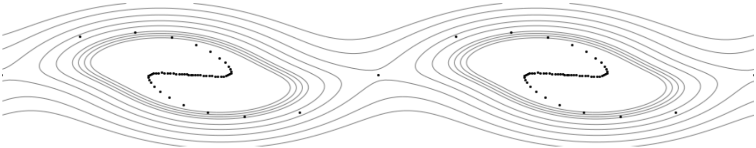
The hyperbolic tangent **mixing layer**

$$\overline{\mathbf{v}}_0 \;=\; U_0 \tanh(z/h)\; \overline{\mathbf{e}}_x$$




displays a **Kelvin-Helmholtz instability** !

Time development: the perturbation $\overline{\mathbf{u}}$ becomes large !



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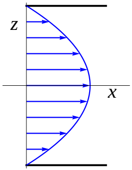
# Stability of inviscid plane Poiseuille flow

**Plane Poiseuille flow** of an **inviscid fluid** has no inflection point $\Rightarrow$ it is **stable**.

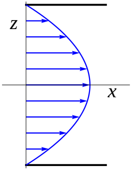
$$\bar{\mathbf{v}}_0 = U_0(1 - (z/h)^2)\,\bar{\mathbf{e}}_x$$



# Stability of viscous plane Poiseuille flow

**Plane Poiseuille flow** of a **viscous fluid** might be **unstable** ?

$$\bar{\mathbf{v}}_0 \;=\; U_0(1-(z/h)^2)\;\bar{\mathbf{e}}_x$$




Must calculate normal modes

$$\psi \;=\; \Psi(z)\;\exp(ikx+\sigma t) \;=\; \Psi(z)\;\exp[ik(x-c_r t)]\;\exp(kc_i t)$$

by solving the **Orr - Sommerfeld equation**

$$\boxed{\sigma D\psi \;=\; -\sigma\Delta\psi \;=\; L_R\psi \;=\; -R^{-1}\Delta\Delta\psi \;+\; ik(U\Delta\psi - U''\psi)}$$

with the BC at $z=\pm 1$ : $\psi \;=\; \partial_z\psi \;=\; 0.$

Eigenvalue $\sigma = -ikc$ ; $c_r = -\sigma_i/k$ phase velocity ;

$\sigma_r > 0 \quad \leftrightarrow \quad$ **amplified mode**

$\sigma_r = 0 \quad \leftrightarrow \quad$ **neutral mode**

$\sigma_r < 0 \quad \leftrightarrow \quad$ **damped mode**

**Exercise 2.2**

# Stability of viscous plane Poiseuille flow: exercise 2.2

$$\boxed{\sigma D\Psi \;=\; -\sigma\Delta\Psi \;=\; L_R\Psi \;=\; -R^{-1}\Delta\Delta\Psi \;+\; ik(U\Delta\Psi - U''\Psi)} \qquad \text{(OS)}$$

$$\text{with} \quad \Delta \;=\; -k^2 \;+\; \frac{d^2}{dz^2}$$

and the boundary conditions $\Psi \;=\; \Psi' \;=\; 0 \quad \text{if} \quad z = \pm 1$ .

**Spectral expansion**

$$\Psi(z) \;=\; \sum_{n=1}^{N} \Psi_n F_n(z)$$

$$\text{with} \quad F_n(z) \;=\; (z-1)^2\,(z+1)^2\;T_{2n-2}(z) \;=\; (z^2-1)^2\;T_{2n-2}(z)\;,$$

$$T_n(z) \;=\; n^{th} \text{ Chebyshev polynomial of the first kind.}$$

Evaluate (OS) at the **Gauss-Lobatto collocation points**

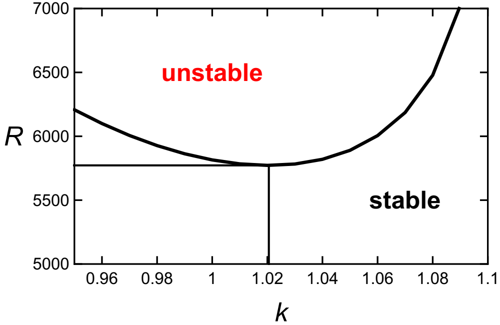
$$z_m \;=\; \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \{1,2,\cdots,N\}$$

$$\iff \quad \sigma\sum_n \Psi_n DF_n(z_m) \;=\; \sum_n \Psi_n LF_n(z_m) \quad \iff \quad \boxed{\sigma MD\cdot V \;=\; ML\cdot V}$$

$$\text{with} \quad V \;=\; (\Psi_1, ..., \Psi_N)^T \quad , \quad MD_{mn} \;=\; DF_n(z_m) \quad , \quad ML_{mn} \;=\; LF_n(z_m)\;.$$

# Stability of viscous plane Poiseuille flow: exercise 2.2

**Neutral curve:**



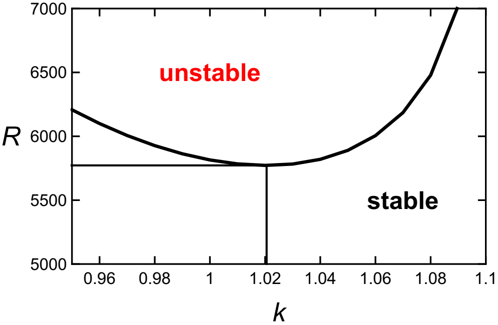

converged, near the critical $k$ corresponding to the minimal $R$, within 0.1% provided that

$$Nz \geq 18$$

which is rather 'low': here the spectral method is quite efficient !

# Stability of viscous plane Poiseuille flow: exercise 2.2

**Neutral curve:**



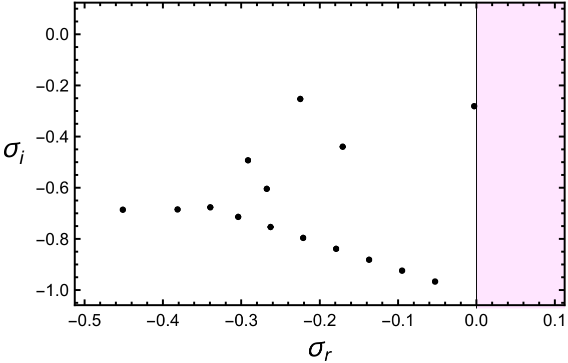

**Patterning bifurcation to traveling 'Tollmienn - Schlichting' waves**

- **critical wavenumber** $k_c = 1.02$
- **critical Reynolds number** $R_c = 5772$
- **critical angular frequency** $\omega_c = 0.269 \leftrightarrow$ **critical phase velocity** $c_c = 0.264$

# Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue $\sigma$ passes the real axis as $R$ increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02 \;, \quad R = 4500$$



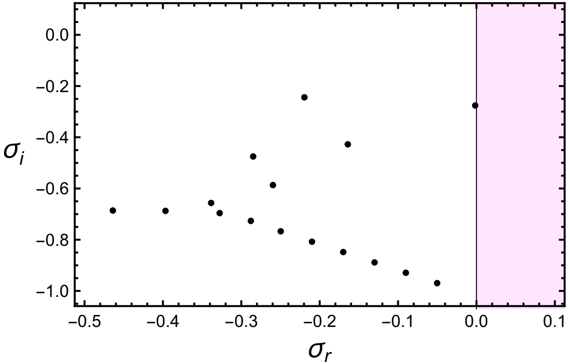

computed with $Nz = 40$ spectral modes and high precision numerics

(collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

# Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue $\sigma$ passes the real axis as $R$ increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02 \; , \quad R = 5000$$



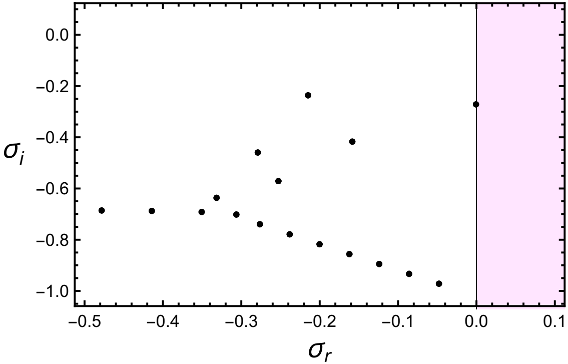

computed with $Nz = 40$ spectral modes and high precision numerics

(collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

# Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue $\sigma$ passes the real axis as $R$ increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02 \;, \quad R = 5500$$



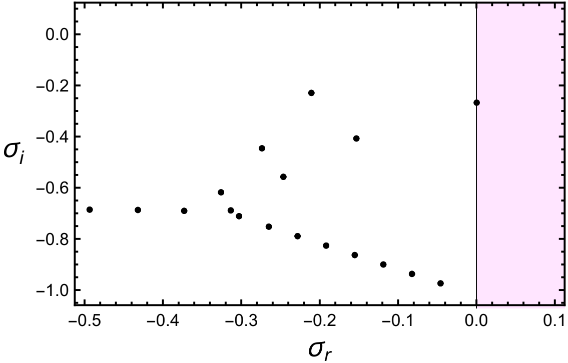

computed with $Nz = 40$ spectral modes and high precision numerics

(collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

# Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue $\sigma$ passes the real axis as $R$ increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02 \ , \quad R = 6000$$



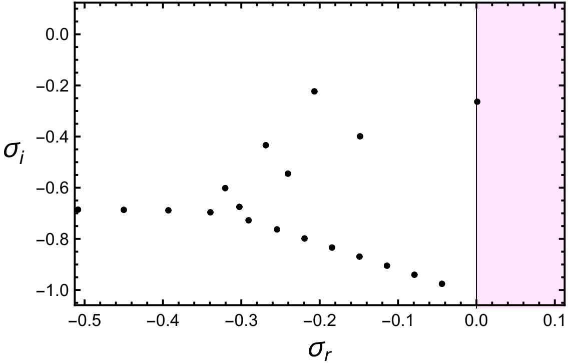

computed with $Nz = 40$ spectral modes and high precision numerics

(collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

# Stability of viscous plane Poiseuille flow

The **bifurcation** corresponds to the fact that one eigenvalue $\sigma$ passes the real axis as $R$ increases, cf. this (part of the) **spectrum of the even modes** for

$$k = 1.02\ ,\quad R = 6500$$




computed with $Nz = 40$ spectral modes and high precision numerics

(collocation points defined with `z[m_]= N[Cos[m Pi/(2 Nz+1)],Nz]`).

# Stability of viscous plane Poiseuille flow: exercise 2.2

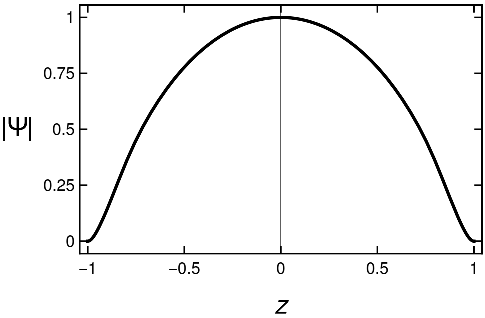
**7** Find the eigenvector of the spectral coefficients

$$V \;=\; (\Psi_1,\, ...,\Psi_N)^T$$

that represents the **critical mode** → calculate the **critical streamfunction**

$$\Psi(z) \;=\; \sum_{n=1}^{N} \Psi_n F_n(z)$$

→ normalize it s.t. $\Psi(z=0) \;=\; 1$ → plot its modulus vs $z$ :



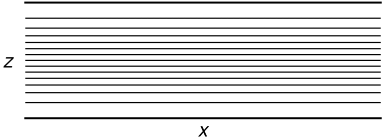
# Stability of viscous plane Poiseuille flow: exercise 2.2

In the $xz$ plane, contour plots of the full streamfunction

$$\Psi_0 \; + \; [A \, \Psi(z) \; \exp(ik_c x) \; + \; c.c.]$$

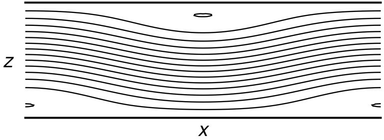
with $\Psi_0$ the one of the basic flow,

for $A = 0$ :



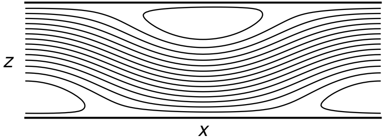

see Reynolds (1895) : motion is ‘direct’

$A = 0.1$ :




motion is ‘sinuous’ !

$A = 0.2$ :




motion is ‘sinuous’ !

# Weakly nonlinear analysis... requires the adjoint problem: ex. 2.3

**1** With the inner product $\langle \psi, \phi \rangle = \int_{x=0}^{\lambda_c} \int_{z=-1}^{1} \psi(x,z)\, \phi^*(x,z)\, \frac{dx}{\lambda_c}\, \frac{dz}{2}$ ,

one can define adjoint operators s.t.

$$\langle D \cdot \psi, \phi \rangle = \left\langle \psi, D^\dagger \cdot \phi \right\rangle \quad \text{and} \quad \langle L \cdot \psi, \phi \rangle = \left\langle \psi, L^\dagger \cdot \phi \right\rangle .$$

For Fourier modes in $x$, of wavenumber $k = mk_c$ with $m \in \mathbb{Z}^*$,

$$D = -\Delta = D^\dagger , \qquad L_R^\dagger \cdot \phi = -R^{-1}\Delta\Delta\phi - 2ikU'\partial_z\phi - ikU\Delta\phi .$$

**2** Code the adjoint problem

$$\sigma^* D \cdot \phi = L_R^\dagger \cdot \phi$$

with the same spectral method as the one for the direct problem.

**3.a** Check: $k = k_c$, $R = R_c$ $\Rightarrow$ $\exists$ adjoint critical mode $\phi_{1c} = \Phi(z)\ \exp(ik_c x)$ corresponding to $\sigma = -i\omega_c$.

**3.b** Calculate $\Phi(z)$, plot $|\Phi(z)|$ and comment.

**4** Normalize $\Phi(z)$ with the condition $\langle D \cdot \psi_{1c}, \phi_{1c} \rangle = 1$ ,

$\psi_{1c} = \Psi(z)\ \exp(ik_c x)$ being the critical mode. Finally, replot $|\Phi(z)|$.