Advanced Fluid Mechanics

Transition to turbulence & turbulence

Applications to thermoconvection, aerodynamics & wind energy

General planning:

Sess° - Man	Date	Content
1 - **EP**	18/12/17	**RB*** Thermoconvection: linear stability analysis
2 - **EP**	08/01/18	**RB** Thermoconvection: weakly nonlinear stability analysis
3 - **EP**	12/01	**RB** Thermoconvection: nonlinear phenomena
→ 4 - **EP**	19/01	Aerodynamics of **OSF***: linear stability analysis
5 - **EP**	22/01	Aerodynamics of **OSF**: linear & weakly nonlinear stability analyses
6 - **EP**	24/01	Aerodynamics of **OSF**: nonlinear phenomena
7 - **JP**	06/02	Wind resources - Conversion principles - Aero
8 - **JP**	07/02	Aero - Wind field and Turbulence
9 - **JP**	08/02	Wind field and Turbulence - Conversion dynamics - Stochastic proc.
JP	08/02	**General conf.: Turbulence & Wind Energy Research**
EP	12/02	**Examination**

RB* = Rayleigh-Bénard **OSF*** = open shear flows

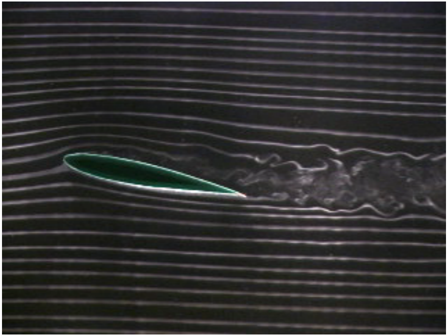
Today: session 4: transition in open shear flows:

- Introduction: instabilities of open shear flows, Rayleigh criterion
- Numerical linear stability analysis of plane Poiseuille flow: towards TS waves

Open shear flows are often encountered in aerodynamics

Turbulent (?) **flow** around an obstacle, an airfoil,

observed with smoke in a wind tunnel at U. Stanford:

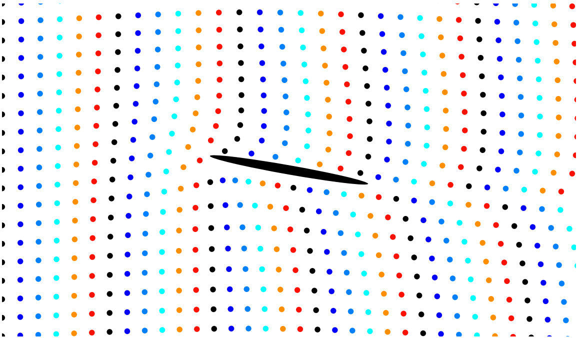


[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Open shear flows are often encountered in aerodynamics

Laminar flow around an obstacle, an airfoil, also exists,

and may be computed with complex analysis techniques:

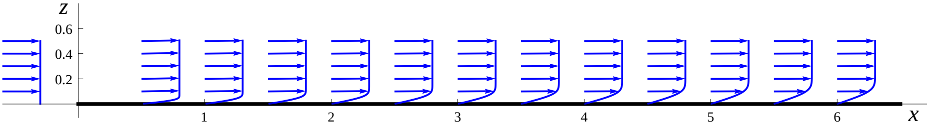


[Plaut E. *Mécanique des fluides.* Cours Mines Nancy 2A]

When and how **laminar open shear flows** get **unstable** and go to **turbulence ?**

When and how 2D xz laminar open shear flows get unstable ?
Example: Blasius boundary layer over a flat plate

Aerodynamical case: x and z in meters: $U = 0.1$ m/s :



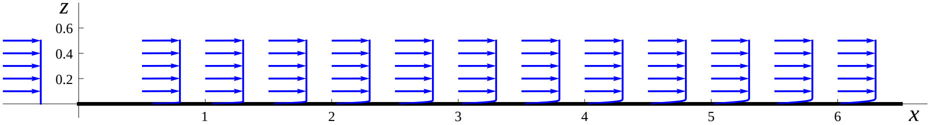
$$\delta = \sqrt{\frac{\nu x}{U}}\ , \quad \zeta = \frac{z}{\delta}\ , \quad v_x = U f'(\zeta)\ , \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\zeta f'(\zeta) - f(\zeta)]$$

Thickness of the boundary layer where $v_x = 0.99U$:

$$\delta_L = 5\sqrt{\frac{\nu x}{U}} \quad \Longleftrightarrow \quad U = 25\,\frac{\nu x}{\delta_L^2} \simeq 25\,\frac{2\ 10^{-5}\ \mathrm{m}^2/\mathrm{s}\ 6\ \mathrm{m}}{0.04\ \mathrm{m}^2} \simeq 0.1\ \mathrm{m/s}$$

When and how 2D xz laminar open shear flows get unstable ?
Example: Blasius boundary layer over a flat plate

Aerodynamical case: x and z in meters: $U = 2$ m/s :



$$\delta = \sqrt{\frac{\nu x}{U}}\,, \quad \zeta = \frac{z}{\delta}\,, \quad v_x = Uf'(\zeta)\,, \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\zeta f'(\zeta) - f(\zeta)]$$

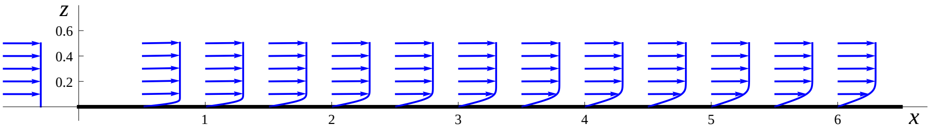
Thickness of the boundary layer where $v_x = 0.99U$:

$$\delta_L = 5\sqrt{\frac{\nu x}{U}}$$

When and how 2D xz laminar open shear flows get unstable ?

Example: Blasius boundary layer over a flat plate

Aerodynamical case: x and z in meters: $U = 0.1$ m/s :



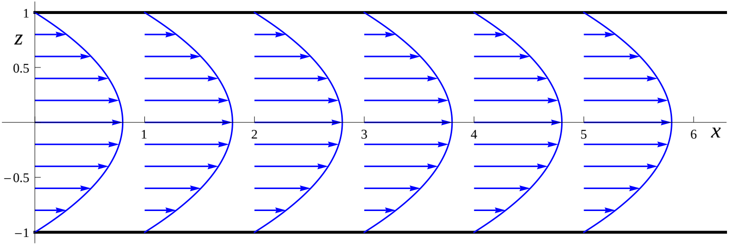
$$\delta = \sqrt{\frac{\nu x}{U}}\,, \quad \zeta = \frac{z}{\delta}\,, \quad v_x = Uf'(\zeta)\,, \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\zeta f'(\zeta) - f(\zeta)]$$

Thickness of the boundary layer where $v_x = 0.99U$:

$$\delta_L = 5\sqrt{\frac{\nu x}{U}}$$

When and how 2D *xz* laminar open shear flows get unstable ?

Example: plane Poiseuille flow



Viscous flow between two plates at $z = \pm h$:

$$\bar{\mathbf{v}} \;=\; U(z)\,\bar{\mathbf{e}}_x \;=\; U_0(1-(z/h)^2)\;,\quad \widehat{p} \;=\; p+\rho g Z \;=-Gx \quad \text{with} \quad G \;=\; 2\eta\frac{U_0}{h^2}\;.$$

Particular case of **plane parallel flow** !

When and how 2D xz laminar open shear flows get unstable ?

General example: plane parallel flows

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 = U(z)\,\overline{\mathbf{e}}_x\ , \quad \widehat{p} = p + \rho g Z = 0 \text{ in an inviscid fluid,}$$

$$\widehat{p} = p + \rho g Z = -Gx \text{ in a viscous fluid,}$$

is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

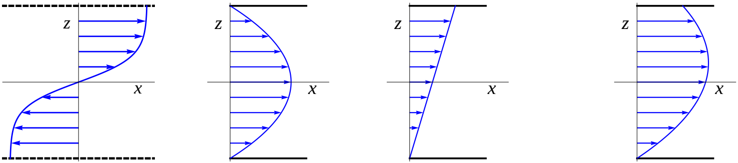
$$\rho\left[\partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}}\cdot\overline{\boldsymbol{\nabla}})\overline{\mathbf{v}}\right] = -\overline{\boldsymbol{\nabla}}\widehat{p} + \eta\overline{\boldsymbol{\Delta}}\overline{\mathbf{v}}$$

$$\Longleftrightarrow \quad \overline{\mathbf{0}} = G\overline{\mathbf{e}}_x + \eta U''(z)\overline{\mathbf{e}}_x$$

whatever $U(z)$ in an inviscid fluid,

provided $U(z) = \alpha + \beta z + \gamma z^2$ in a viscous fluid.

Mixing layer — **Poiseuille flow** — **Couette flow** — **Couette-Poiseuille flow**



inviscid fl. — viscous fl. — viscous fl. — viscous fl.

Stability analysis of plane parallel flows

Basic flow:

$$\overline{\mathbf{v}}_0 \;=\; U(z)\,\overline{\mathbf{e}}_x\,,\quad \widehat{p}_0 \;=\; p_0+\rho g Z \;=-Gx \quad \text{with} \quad G=0 \text{ in an inviscid fluid},$$
$$G>0 \text{ in a viscous fluid.}$$

Basic flow with **perturbations**:

$$\overline{\mathbf{v}} \;=\; \overline{\mathbf{v}}_0 \;+\; \overline{\mathbf{u}}\,,\quad \widehat{p} \;=\; \widehat{p}_0 \;+\; p'$$

$$\partial_t\overline{\mathbf{v}} \;+\; (\overline{\mathbf{v}}\cdot\overline{\boldsymbol{\nabla}})\overline{\mathbf{v}} \;=\; -(1/\rho)\overline{\boldsymbol{\nabla}}\widehat{p} \;+\; \nu\overline{\boldsymbol{\Delta}}\overline{\mathbf{v}} \qquad \text{(NS)}$$

$$\partial_t\overline{\mathbf{u}} \;+\; U' u_z\overline{\mathbf{e}}_x \;+\; U\partial_x\overline{\mathbf{u}} \;+\; (\overline{\mathbf{u}}\cdot\overline{\boldsymbol{\nabla}})\overline{\mathbf{u}} \;=\; -(1/\rho)\overline{\boldsymbol{\nabla}}p' \;+\; \nu\overline{\boldsymbol{\Delta}}\overline{\mathbf{u}} \qquad \text{(NS)}$$

$$\mathrm{div}\overline{\mathbf{v}} \;=\; 0 \qquad \text{(MC)}$$

$$\mathrm{div}\overline{\mathbf{u}} \;=\; 0 \qquad \text{(MC)}$$

- ▷ Unit of length $=\; h$ half-width of the channel, thickness of the mixing layer...
- ▷ Unit of velocity $=\; U_0 \;=\; \max_z U(z)$ scale of U
- ▷ Unit of time $=\; h/U_0$ advection time

$$\partial_t\overline{\mathbf{u}} \;+\; U' u_z\overline{\mathbf{e}}_x \;+\; U\partial_x\overline{\mathbf{u}} \;+\; (\overline{\mathbf{u}}\cdot\overline{\boldsymbol{\nabla}})\overline{\mathbf{u}} \;=\; -\overline{\boldsymbol{\nabla}}p'' \;+\; R^{-1}\,\overline{\boldsymbol{\Delta}}\overline{\mathbf{u}} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R \;=\; U_0 h/\nu}$, $R \;=\; \infty$ in an inviscid fluid.

2D *xz* stability analysis of plane parallel flows

Dimensionless equations for the **perturbations** $\overline{\mathbf{u}}$ of velocity and p'' of pressure:

$$\partial_t \overline{\mathbf{u}} \;+\; U' u_z \overline{\mathbf{e}}_x \;+\; U \partial_x \overline{\mathbf{u}} \;+\; (\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}}) \overline{\mathbf{u}} \;=\; -\overline{\boldsymbol{\nabla}} p'' \;+\; R^{-1}\, \overline{\boldsymbol{\Delta}} \overline{\mathbf{u}} \;, \qquad \text{(NS)}$$

$$\mathrm{div} \overline{\mathbf{u}} \;=\; 0 \;. \qquad \text{(MC)}$$

2D *xz* **perturbations** can be defined by their **streamfunction** $\psi(x,z)$:

$$\overline{\mathbf{u}} \;=\; \overline{\mathbf{curl}}(\psi\; \overline{\mathbf{e}}_y) \;=\; (\overline{\boldsymbol{\nabla}} \psi) \times \overline{\mathbf{e}}_y \;=\; -(\partial_z \psi)\; \overline{\mathbf{e}}_x \;+\; (\partial_x \psi)\; \overline{\mathbf{e}}_z \;.$$

How can one eliminate p'' in (NS) ? Consider $\overline{\mathbf{curl}}(\mathrm{NS}) \cdot \overline{\mathbf{e}}_y$ i.e. the **vorticity equation**:

$$\partial_t(-\Delta\psi) + \left[\partial_z\left(\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_x\right) - \partial_x\left(\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_z\right)\right] \;=\; R^{-1} \Delta(-\Delta\psi) + U \partial_x(\Delta\psi) - U''(\partial_x\psi) \;. \qquad \text{(Vort)}$$

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi \;+\; N_2(\psi,\psi)} \;. \qquad \text{(Vort)}$$

Boundary conditions:

viscous fluid : $\overline{\mathbf{u}} = \overline{\mathbf{0}} \iff \partial_x \psi \;=\; \partial_z \psi \;=\; 0 \quad \text{if} \quad z = z_\pm \;,$

inviscid fluid : $u_z = 0 \iff \partial_x \psi \;=\; 0 \quad \text{if} \quad z = z_\pm \;.$

2D xz linear stability analysis of plane parallel flows

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \;=\; -\Delta \partial_t \psi \;, \quad L_R \cdot \psi \;=\; R^{-1} \Delta(-\Delta\psi) \;+\; U \partial_x(\Delta\psi) - U''(\partial_x \psi) \;,$$

$$\text{viscous fluid :} \quad \overline{\mathbf{u}} = \overline{\mathbf{0}} \quad \Longleftrightarrow \quad \partial_x \psi \;=\; \partial_z \psi \;=\; 0 \quad \text{if} \quad z = z_\pm \;,$$

$$\text{inviscid fluid :} \quad u_z = 0 \quad \Longleftrightarrow \quad \partial_x \psi \;=\; 0 \quad \text{if} \quad z = z_\pm \;.$$

Normal mode analysis:

$$\psi \;=\; \Psi_n(z)\; \exp(ikx + \sigma t) \;=\; \Psi_n(z)\; \exp[ik(x - c_r t)]\; \exp(kc_i t)$$

with k = **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
σ = **temporal eigenvalue**.

Most often the bulk velocity of the basic flow $\langle U \rangle_z > 0 \;\Rightarrow$ by advection

$\sigma \;=\; -i\omega \;=\; -ikc$ with c the **complex phase velocity**,
$c_r > 0$ the **real phase velocity**,
$kc_i > 0$ (resp. < 0) the **growth rate** (resp. damping rate).

2D xz linear stability analysis of plane parallel flows

$$-\sigma\Delta\psi = R^{-1}\Delta(-\Delta\psi) + U\partial_x(\Delta\psi) - U''(\partial_x\psi) \quad \text{(Vort)}$$

$$\Longleftrightarrow \quad ikc\Delta\psi = R^{-1}\Delta(-\Delta\psi) + ikU\Delta\psi - ikU''\psi \quad \text{(Vort)}$$

$$\Longleftrightarrow \quad \boxed{(U-c)\Delta\psi - U''\psi = (ikR)^{-1}\Delta\Delta\psi} \quad \text{(Vort)}$$

Orr - Sommerfeld eq. in a viscous fluid, **Rayleigh eq.** in an inviscid fluid ($R = \infty$)

BC at $z = z_\pm$: viscous fluid: $\psi = \partial_z\psi = 0$; inviscid fluid: $\psi = 0$.

Normal mode analysis:

$$\psi = \Psi_n(z)\ \exp(ikx + \sigma t) = \Psi_n(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

with k = **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
σ = **temporal eigenvalue**.

Most often the bulk velocity of the basic flow $\langle U\rangle_z > 0 \quad \Rightarrow$ by advection

$\sigma = -i\omega = -ikc$ with c the **complex phase velocity**,
$c_r > 0$ the **real phase velocity**,

$kc_i > 0$ (resp. < 0) the **growth rate** (resp. damping rate).

2D *xz* linear stability analysis of inviscid plane parallel flows

Normal mode analysis: assume there is at least one **amplified mode**

$$\psi = \Psi(z)\ \exp(ikx - ikct) = \Psi(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

with c_r the **real phase velocity**, $kc_i > 0$ the **growth rate**.

It satisfies the **Rayleigh equation** $\boxed{(U - c)\Delta\psi - U''\psi = 0}$

with the BC $\psi = 0$ if $z = z_\pm$.

Exercise 2.1 **Rayleigh's inflection point criterion**

▷ Express $\Psi''(z)$ as a function of $\Psi(z)$, $U(z)$, $U''(z)$, k and c.

▷ By multiplication with a suitable function and integration over $z \in [z_-, z_+]$, show that

$$\int_{z_-}^{z_+} \left(k^2|\Psi(z)|^2 + |\Psi'(z)|^2\right) dz + \int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{U(z) - c} dz = 0$$

and

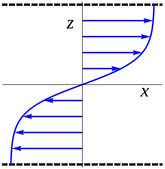
$$\int_{z_-}^{z_+} \frac{U''(z)\ |\Psi(z)|^2}{|U(z) - c|^2} dz = 0 \Rightarrow \text{ if } U'' \neq 0\ ,\ U'' \text{ must change sign somewhere,}$$

there must exist an **inflection point** in the *U*-profile.

Instability of an inviscid plane parallel flow, the mixing layer

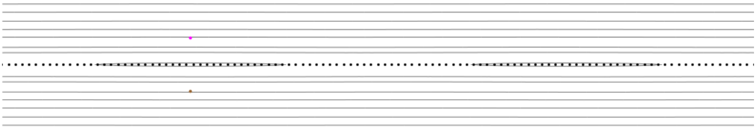
The hyperbolic tangent **mixing layer**

$$\bar{\mathbf{v}}_0 = U_0 \tanh(z/h)\, \bar{\mathbf{e}}_x$$



displays a **Kelvin-Helmholtz Instability** !

Initial condition $\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 + \bar{\mathbf{u}}$ with $\bar{\mathbf{u}}$ small:

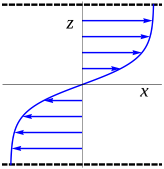


[Plaut E. *Mécanique des fluides.* Cours Mines Nancy 2A]

Instability of an inviscid plane parallel flow, the mixing layer

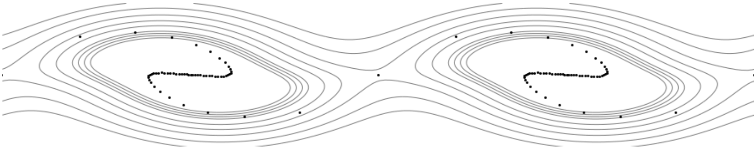
The hyperbolic tangent **mixing layer**

$$\bar{\mathbf{v}}_0 = U_0 \tanh(z/h)\, \bar{\mathbf{e}}_x$$



displays a **Kelvin-Helmholtz instability** !

Time development: the perturbation $\bar{\mathbf{u}}$ becomes large !

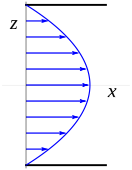


[Plaut E. *Mécanique des fluides.* Cours Mines Nancy 2A]

Stability of inviscid plane Poiseuille flow

Plane Poiseuille flow of an **inviscid fluid** has no inflection point $\Rightarrow$ it is **stable**.

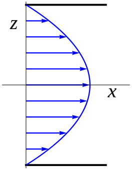
$$\overline{\mathbf{v}}_0 = U_0(1-(z/h)^2)\ \overline{\mathbf{e}}_x$$



Stability of viscous plane Poiseuille flow

Plane Poiseuille flow of a **viscous fluid** might be **unstable** ?

$$\bar{\mathbf{v}}_0 = U_0(1 - (z/h)^2)\ \bar{\mathbf{e}}_x$$



Must calculate normal modes

$$\psi = \Psi(z)\ \exp(ikx + \sigma t) = \Psi(z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

by solving the **Orr - Sommerfeld equation**

$$\boxed{\sigma D\psi = -\sigma\Delta\psi = L_R\psi = -R^{-1}\Delta\Delta\psi + ik(U\Delta\psi - U''\psi)}$$

with the BC at $z = \pm 1$: $\psi = \partial_z\psi = 0.$

Eigenvalue $\sigma = -ikc$; $c_r = -\sigma_i/k$ phase velocity ;

$\sigma_r > 0 \quad \leftrightarrow$ **amplified mode**

$\sigma_r = 0 \quad \leftrightarrow$ **neutral mode**

$\sigma_r < 0 \quad \leftrightarrow$ **damped mode**

Exercise 2.2

Stability of viscous plane Poiseuille flow: exercise 2.2

$$\sigma D\Psi = -\sigma\Delta\Psi = L_R\Psi = -R^{-1}\Delta\Delta\Psi + ik(U\Delta\Psi - U''\Psi) \qquad \text{(OS)}$$

$$\text{with} \quad \Delta = -k^2 + \frac{d^2}{dz^2}$$

and the boundary conditions $\Psi = \Psi' = 0$ if $z = \pm 1$.

Spectral expansion

$$\Psi(z) = \sum_{n=1}^{N} \Psi_n F_n(z)$$

$$\text{with} \quad F_n(z) = (z-1)^2 (z+1)^2 \, T_{2n-2}(z) = (z^2-1)^2 \, T_{2n-2}(z) \, ,$$

$$T_n(z) = n^{th} \text{ Chebyshev polynomial of the first kind.}$$

Evaluate (OS) at the **Gauss-Lobatto collocation points**

$$z_m = \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \{1,2,\cdots,N\}$$

$$\Longleftrightarrow \quad \sigma\sum_n \Psi_n DF_n(z_m) = \sum_n \Psi_n LF_n(z_m) \quad \Longleftrightarrow \quad \sigma MD\cdot V = ML\cdot V$$

$$\text{with} \quad V = (\Psi_1, ..., \Psi_N)^T \quad , \quad MD_{mn} = DF_n(z_m) \quad , \quad ML_{mn} = LF_n(z_m) \, .$$