Advanced Fluid Mechanics
Transition to Turbulence & Turbulence
Applications to Transfers, Aerodynamics & Wind Energy

General planning:

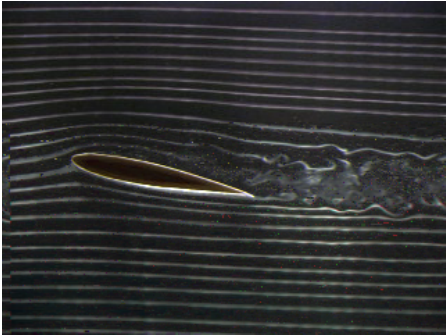
Sess^{o} - **Man**	Date	Content
1 - **EP**	09/01 m	Thermoconvection with slip boundary conditions
2 - **EP**	11/01 m	Transfers in thermoconvection with various boundary conditions
		Patterning, supercritical bifurcations
3 - **JP**	17/01 m	Wind resources - Conversion principles - Aero
4 - **JP**	18/01 m	Aero - Wind eld and Turbulence
5 - **JP**	19/01 m	Wind eld and Turbulence - Conversion dynamics - Stochastic proc.
JP	19/01 a	**General conf.: Turbulence & Wind Energy Research**
→ 6 - **EP**	23/01 m	Aerodynamics: linear stability of an open shear ow
7 - **EP**	25/01 m	Aerodynamics: nonlinear stability of open shear ows
		Patterning, subcritical & saddle-node bifurcations
EP	10/02 a	**Examination**

Today: Session 6: Transition in Open Shear Flows:

- Introduction: Instabilities of Open Shear Flows, Rayleigh criterion
- Numerical linear stability analysis of Plane Poiseuille Flow: TS waves

Open Shear Flows are often encountered in Aerodynamics

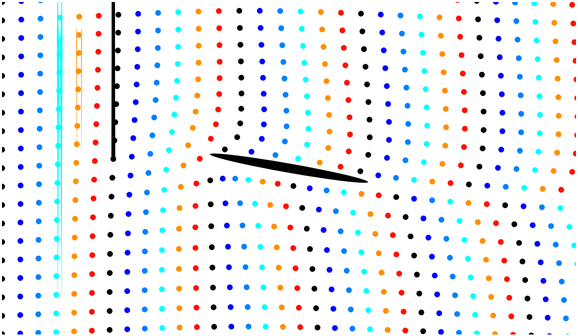
Turbulent Flow around an obstacle, an Airfoil,
observed with smoke in a Wind Tunnel at U. Stanford:



[DVD `Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Open Shear Flows are often encountered in Aerodynamics

Laminar Flow around an obstacle, an Airfoil, also exists,
and may be computed with Complex Analysis Techniques:



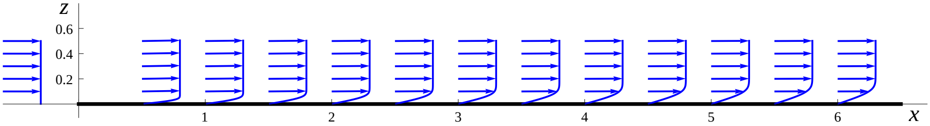
[Plaut E. *Mecanique des uides*. Cours Mines Nancy 2A]

When and how **Laminar Open Shear Flows** get **unstable** and go to **Turbulence ?**

When and how 2D xz Laminar Open Shear Flows get unstable ?

Example: Blasius Boundary Layer over a flat plate

Aerodynamical case: x and z in meters: $U = 0.1$ m/s :



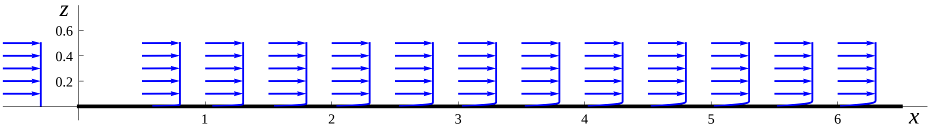
$$\delta = \sqrt{\frac{\nu x}{U}}\ , \quad \zeta = \frac{z}{\delta}\ , \quad v_x = U f'(\zeta)\ , \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\zeta f'(\zeta) - f(\zeta)\right]$$

Thickness of the Boundary Layer where $v_x = 0.99U$:

$$\delta_L = 5\sqrt{\frac{\nu x}{U}} \iff U = 25\,\frac{\nu x}{\delta_L^2} \simeq 25\,\frac{2\,10^{-5}\ \mathrm{m^2/s}\ 6\ \mathrm{m}}{0.04\ \mathrm{m^2}} \simeq 0.1\ \mathrm{m/s}$$

When and how 2D xz Laminar Open Shear Flows get unstable ?
Example: Blasius Boundary Layer over a flat plate

Aerodynamical case: x and z in meters: $U = 2$ m/s :



$$\delta = \sqrt{\frac{\nu x}{U}} \, , \quad \zeta = \frac{z}{\delta} \, , \quad v_x = U f'(\zeta) \, , \quad v_z = \frac{1}{2} \sqrt{\frac{\nu U}{x}} [\zeta f'(\zeta) - f(\zeta)]$$

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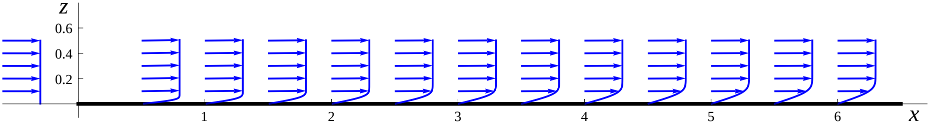

$$\delta = \sqrt{\frac{\nu x}{U}} , \quad \zeta = \frac{z}{\delta} , \quad v_x = U f'(\zeta) , \quad v_z = \frac{1}{2}\sqrt{\frac{\nu U}{x}} [\zeta f'(\zeta) - f(\zeta)]$$

Thickness of the Boundary Layer where $v_x = 0.99U$:

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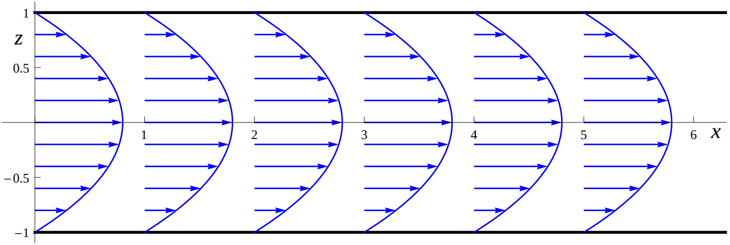
When and how 2D xz Laminar Open Shear Flows get unstable ?

Example: Plane Poiseuille Flow



Viscous ow between two plates at $z = \pm h$:

$$\bar{\mathbf{v}} = U(z)\,\bar{\mathbf{e}}_x = U_0\left(1-(z/h)^2\right)\,, \quad \hat{p} = p + \rho g Z = -Gx \quad \text{with} \quad G = 2\eta\frac{U_0}{h^2}\,.$$

Particular case of **Plane Parallel Flow** !

When and how 2D xz Laminar Open Shear Flows get unstable ?

General Example: Plane Parallel Flows

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 = U(z)\,\overline{\mathbf{e}}_x\ , \quad \text{þ} = p + \rho g Z = 0 \quad \text{in an inviscid uid,}$$

$$\text{þ} = p + \rho g Z = -Gx \quad \text{in a viscous uid,}$$

is solution of the Euler ($\eta = 0$) of Navier-Stokes ($\eta \neq 0$) equation

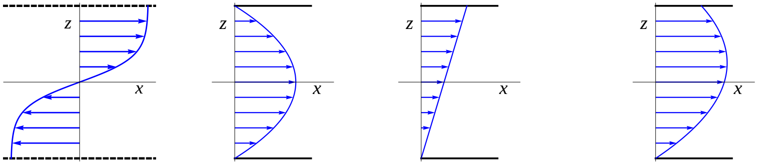
$$\rho\ \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \overline{\boldsymbol{\nabla}})\overline{\mathbf{v}} = -\overline{\boldsymbol{\nabla}}\text{þ} + \eta \overline{\boldsymbol{\Delta}}\overline{\mathbf{v}}$$

$$\Longleftrightarrow \quad \overline{\mathbf{0}} = G\overline{\mathbf{e}}_x + \eta U^{00}(z)\overline{\mathbf{e}}_x$$

whatever $U(z)$ in an inviscid uid,

provided $U(z) = \alpha + \beta z + \gamma z^2$ in a viscous uid.

Mixing Layer | Poiseuille Flow | Couette Flow | Couette-Poiseuille Flow



inviscid . | viscous . | viscous . | viscous .

Stability Analysis of Plane Parallel Flows

Basic ow:

$$\bar{\mathbf{v}}_0 = U(z)\,\bar{\mathbf{e}}_x\,, \quad \not{p}_0 = p_0 + \rho g Z = -Gx \quad \text{with} \quad G = 0 \text{ in an inviscid uid,}$$

$$G > 0 \text{ in a viscous uid.}$$

Basic ow with **perturbations**:

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 + \bar{\mathbf{u}}\,, \quad \not{p} = \not{p}_0 + p^{0}$$

$$\partial_t \bar{\mathbf{v}} + (\bar{\mathbf{v}} \cdot \overline{\boldsymbol{\nabla}})\bar{\mathbf{v}} = -(1/\rho)\overline{\boldsymbol{\nabla}}\not{p} + \nu \overline{\boldsymbol{\Delta}}\bar{\mathbf{v}} \qquad \text{(NS)}$$

$$\partial_t \bar{\mathbf{u}} + U^{0} u_z \bar{\mathbf{e}}_x + U\partial_x \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}})\bar{\mathbf{u}} = -(1/\rho)\overline{\boldsymbol{\nabla}} p^{0} + \nu \overline{\boldsymbol{\Delta}}\bar{\mathbf{u}} \qquad \text{(NS)}$$

$$\mathrm{div}\bar{\mathbf{v}} = 0 \qquad \text{(MC)}$$

$$\mathrm{div}\bar{\mathbf{u}} = 0 \qquad \text{(MC)}$$

- ▷ Unit of length = h half-width of the channel, thickness of the mixing layer...
- ▷ Unit of velocity = $U_0 = \max_z U(z)$ scale of U
- ▷ Unit of time = h/U_0 advection time

$$\partial_t \bar{\mathbf{u}} + U^{0} u_z \bar{\mathbf{e}}_x + U\partial_x \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}})\bar{\mathbf{u}} = -\overline{\boldsymbol{\nabla}} p^{00} + R^{\ 1}\, \overline{\boldsymbol{\Delta}}\bar{\mathbf{u}} \qquad \text{(NS)}$$

with the **Reynolds number** $\boxed{R = U_0 h/\nu}$, $R = \infty$ in an inviscid uid.

2D *xz* Stability Analysis of Plane Parallel Flows

Dimensionless equations for the **perturbations** $\overline{\mathbf{u}}$ of velocity and p^{00} of pressure:

$$\partial_t \overline{\mathbf{u}} \; + \; U^0 u_z \overline{\mathbf{e}}_x \; + \; U \partial_x \overline{\mathbf{u}} \; + \; (\overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}}) \overline{\mathbf{u}} \; = \; - \overline{\boldsymbol{\nabla}} p^{00} \; + \; R \;\; ^1 \; \overline{\boldsymbol{\Delta}} \overline{\mathbf{u}} \; , \qquad \text{(NS)}$$

$$\mathrm{div} \overline{\mathbf{u}} \; = \; 0 \; . \qquad \text{(MC)}$$

2D *xz* **perturbations** can be de ned by their **streamfunction** $\psi(x,z)$:

$$\overline{\mathbf{u}} \; = \; \overline{\mathbf{curl}}(\psi \; \overline{\mathbf{e}}_y) \; = \; (\overline{\boldsymbol{\nabla}} \psi) \times \overline{\mathbf{e}}_y \; = \; - (\partial_z \psi) \; \overline{\mathbf{e}}_x \; + \; (\partial_x \psi) \; \overline{\mathbf{e}}_z \; .$$

How can one eliminate p^{00} in (NS) ? Consider $\overline{\mathbf{curl}}(\text{NS}) \cdot \overline{\mathbf{e}}_y$ i.e. the **vorticity equation**:

$$\partial_t (- \;\; \psi) + \; \partial_z \; \overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_x \; - \partial_x \; \overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} u_z \; = R \;\; ^1 \; (- \;\; \psi) + U \partial_x (\;\; \psi) - U^{00} (\partial_x \psi) \; . \qquad \text{(Vort)}$$

$$\boxed{D \cdot \partial_t \psi \; = \; L_R \cdot \psi \; + \; N_2(\psi,\psi)} \; . \qquad \text{(Vort)}$$

Boundary conditions:

Viscous uid : $\overline{\mathbf{u}} = \overline{\mathbf{0}} \iff \partial_x \psi = \partial_z \psi = 0$ if $z = z$,

Inviscid uid : $u_z = 0 \iff \partial_x \psi = 0$ if $z = z$.

2D xz Linear Stability Analysis of Plane Parallel Flows

$$\boxed{D \cdot \partial_t \psi \;=\; L_R \cdot \psi} \qquad \text{(Vort)}$$

$$D \cdot \partial_t \psi \;=\; - \quad \partial_t \psi \;, \quad L_R \cdot \psi \;=\; R \;\; ^{1} \;\; (- \quad \psi) \;+\; U\partial_x(\quad \psi) - U^{00}(\partial_x \psi) \;,$$

$$\begin{array}{llll} \text{Viscous} & \text{uid :} \quad \bar{\mathbf{u}} = \bar{\mathbf{0}} & \Longleftrightarrow \quad \partial_x \psi \;=\; \partial_z \psi \;=\; 0 & \text{if} \quad z = z \quad , \\ \text{Inviscid} & \text{uid :} \quad u_z = 0 & \Longleftrightarrow \quad \partial_x \psi \;=\; 0 & \text{if} \quad z = z \quad . \end{array}$$

Normal mode analysis:

$$\psi \;=\; \quad _n(z) \;\exp(ikx + \sigma t) \;=\; \quad _n(z) \;\exp[ik(x - c_r t)] \;\exp(kc_i t)$$

with k = **horizontal wavenumber**, $k \neq 0$, n another label to mark normal modes,
σ = **temporal eigenvalue**.

Most often the bulk velocity of the basic ow $\langle U \rangle_z > 0 \quad \Rightarrow$ by advection

$\sigma \;=\; - i\omega \;=\; - ikc$ with c the **complex phase velocity**,
$c_r > 0$ the **real phase velocity**,
$kc_i > 0$ (resp. < 0) the **growth rate** (resp. damping rate).

2D xz Linear Stability Analysis of Plane Parallel Flows

$$-\sigma \quad \psi \;=\; R \;\; ^{1} \;\; (- \quad \psi) \;+\; U\partial_x(\quad \psi) - U^{00}(\partial_x \psi) \qquad \text{(Vort)}$$

$$\iff \quad ikc \quad \psi \;=\; R \;\; ^{1} \;\; (- \quad \psi) \;+\; ikU \quad \psi - ikU^{00}\psi \qquad \text{(Vort)}$$

$$\iff \quad \boxed{(U - c) \quad \psi \;-\; U^{00}\psi \;=\; (ikR) \;\; ^{1} \quad\quad \psi} \qquad \text{(Vort)}$$

Orr - Sommerfeld eq. in a viscous uid, **Rayleigh eq.** in an inviscid uid ($R = \infty$)

B.C. at $z = z$: Viscous uid: $\psi \;=\; \partial_z \psi \;=\; 0$; Inviscid uid: $\psi \;=\; 0$.

Normal mode analysis:

$$\psi \;=\; \quad _n(z) \; \exp(ikx + \sigma t) \;=\; \quad _n(z) \; \exp[ik(x - c_r t)] \; \exp(kc_i t)$$

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2D xz Linear Stability Analysis of Inviscid Plane Parallel Flows

Normal mode analysis: assume there is at least one **amplified mode**

$$\psi = (z)\ \exp(ikx - ikct) = (z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

with c_r the **real phase velocity**, $kc_i > 0$ the **growth rate**.

It satis es the **Rayleigh equation** $(U - c)\ \psi - U^{00}\psi = 0$

with the B.C. $\psi = 0$ if $z = z$.

Exercise 2.1 **Rayleigh's inflection point criterion**

▷ Express $^{00}(z)$ as a function of (z), $U(z)$, $U^{00}(z)$, k and c.

▷ By multiplication with a suitable function and integration over $z \in [z\ , z_+]$, show that

$$\int_{z_-}^{z_+} k^2 |\ (z)|^2 + |\ ^0(z)|^2\ dz + \int_{z_-}^{z_+} \frac{U^{00}(z)\ |\ (z)|^2}{U(z) - c} dz = 0$$

and

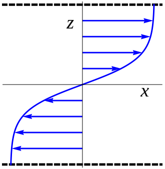
$$\int_{z_-}^{z_+} \frac{U^{00}(z)\ |\ (z)|^2}{|U(z) - c|^2} dz = 0 \Rightarrow \text{ if } U^{00} \neq 0\ ,\ U^{00} \text{ must change sign somewhere,}$$

there must exist an **inflection point** in the U-pro le.

Instability of an Inviscid Plane Parallel Flow, the Mixing Layer

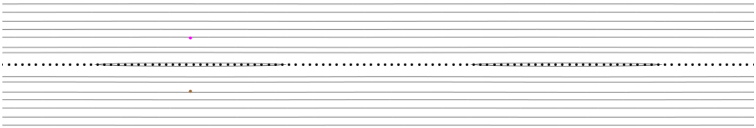
The Hyperbolic Tangent **Mixing Layer**

$$\bar{\mathbf{v}}_0 = U_0 \tanh(z/h)\ \bar{\mathbf{e}}_x$$



displays a **Kelvin-Helmholtz Instability** !

Initial condition $\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 + \bar{\mathbf{u}}$ with $\bar{\mathbf{u}}$ small:



[Plaut E. *Mecanique des uides*. Cours Mines Nancy 2A]

Instability of an Inviscid Plane Parallel Flow, the Mixing Layer

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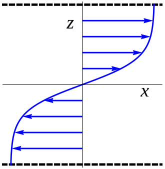
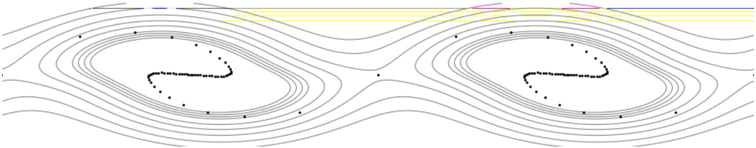


displays a **Kelvin-Helmholtz Instability** !

Time development: the perturbation $\overline{\mathbf{u}}$ becomes large !



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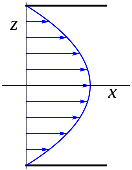
Stability of Inviscid Plane Poiseuille Flow

Plane Poiseuille Flow of an **Inviscid Fluid** has no in ection point $\Rightarrow$ it is **stable**.

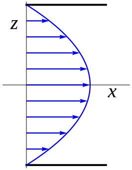
$$\bar{\mathbf{v}}_0 = U_0(1-(z/h)^2)\,\bar{\mathbf{e}}_x$$



Stability of Viscous Plane Poiseuille Flow

Plane Poiseuille Flow of a **Viscous Fluid** might be **unstable** ?

$$\bar{\mathbf{v}}_0 = U_0(1 - (z/h)^2)\ \bar{\mathbf{e}}_x$$



Must calculate normal modes

$$\psi = \ (z)\ \exp(ikx + \sigma t) = \ (z)\ \exp[ik(x - c_r t)]\ \exp(kc_i t)$$

by solving the **Orr - Sommerfeld equation**

$$\boxed{\sigma D\psi = -\sigma \quad \psi = L_R\psi = -R^{\ 1} \quad \psi + ik(U \quad \psi - U^{00}\psi)}$$

with the B.C. at $z = \pm 1$: $\psi = \partial_z\psi = 0.$

Eigenvalue $\sigma = -ikc$; $c_r = -\sigma_i/k$ phase velocity ;

$\sigma_r > 0 \leftrightarrow$ **amplified mode**

$\sigma_r = 0 \leftrightarrow$ **neutral mode**

$\sigma_r < 0 \leftrightarrow$ **damped mode**

Exercise 2.2

Stability of Viscous Plane Poiseuille Flow: Ex. 2.2

$$\boxed{\sigma D \quad = \quad -\sigma \quad = \quad L_R \quad = \quad -R^{\;1} \quad + \quad ik(U \quad - U^{00} \quad)} \qquad \text{(OS)}$$

with

$$= \quad -k^2 \quad + \quad \frac{d^2}{dz^2}$$

and the boundary conditions $= \quad {}^{0} \quad = \quad 0 \quad \text{if} \quad z = \pm 1 \; .$

Spectral expansion

$$(z) \quad = \quad \sum_{n=1}^{N} \quad {}_n F_n(z)$$

with

$$F_n(z) \quad = \quad (z-1)^2 \; (z+1)^2 \; T_{2n \quad 2}(z) \quad = \quad (z^2-1)^2 \; T_{2n \quad 2}(z)$$

Evaluate (OS) at the **Gauss-Lobatto collocation points**

$$z_m \quad = \quad \cos[m\pi/(2N+1)] \quad \text{for} \quad m \in \{1,2,\cdots,N\}$$

$$\Longleftrightarrow \quad \sigma \sum_n \quad {}_n DF_n(z_m) \quad = \quad \sum_n \quad {}_n LF_n(z_m) \quad \Longleftrightarrow \quad \sigma MD \cdot V \quad = \quad ML \cdot V$$

$$\text{with} \quad V \quad = \quad (\quad {}_1, \ldots, \quad {}_N)^T \quad , \quad MD_{mn} \quad = \quad DF_n(z_m) \quad , \quad ML_{mn} \quad = \quad LF_n(z_m) \; .$$